

ON A CERTAIN SENARY CUBIC FORM

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ABSTRACT. A strong form of the Manin-Peyre conjecture with a power saving error term is proved for a certain cubic fourfold.

1. INTRODUCTION

1.1. Principal results. The main goal of this paper is the verification of predictions due to Manin and Peyre concerning the distribution of rational points on the cubic fourfold in \mathbb{P}^5 defined by the equation

$$(1.1) \quad W : \quad x_1 y_2 y_3 + x_2 y_1 y_3 + x_3 y_1 y_2 = 0.$$

We shall derive an asymptotic formula not only in line with the aforementioned expectation, but of strength sufficient to obtain an analytic continuation for the associated height zeta function beyond the region of absolute convergence. Along the way, we construct a crepant resolution of its singularities and determine the universal torsor, thus providing a comprehensive picture of the arithmetic and algebraic properties of the fourfold defined by (1.1). The properties of this fourfold are sufficiently distinct from those among the small stock of cubic fourfolds for which the Manin-Peyre conjectures are already known, to require treatment by a new analytic toolbox.

More notation is required for precise statements of our results as well as comments on the methods involved. If a point in \mathbb{P}^5 is represented by $(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{Z}^6$ with coprime coordinates, then $H(\mathbf{x}, \mathbf{y}) = \max(|x_j|, |y_j|)^3$ is a natural anticanonical height function on $W(\mathbb{Q})$. The coordinate 3-planes where at least one of y_1, y_2, y_3 vanishes are accumulating subsets of W with exceptionally many points. For instance, choosing $y_1 = y_2 = 0$ and integral x_1, x_2, x_3, y_3 subject only to a coprimality condition, we find about $P^{4/3}$ rational points $(\mathbf{x}, \mathbf{y}) \in \mathbb{P}^5$ satisfying the equations (1.1) and $y_1 y_2 y_3 = 0$ as well as the height condition

$$(1.2) \quad H(\mathbf{x}, \mathbf{y}) \leq P.$$

Our principal result concerns the density of the rational points on the Zariski open subset W° defined by (1.1) and

$$(1.3) \quad y_1 y_2 y_3 \neq 0.$$

Let $N(P)$ denote the number of rational points on W° satisfying satisfying (1.2).

Theorem 1. *There is a constant $\delta > 0$ and a real polynomial Q of degree 4 such that*

$$N(P) = P Q(\log P) + O(P^{1-\delta}).$$

The leading coefficient of Q equals

$$(1.4) \quad \frac{1}{324}(\pi^2 + 24 \log 2 - 3) \prod_p \left(1 - \frac{1}{p}\right)^5 \left(1 + \frac{5}{p} + \frac{6}{p^2} + \frac{5}{p^3} + \frac{1}{p^4}\right).$$

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It is possible to provide a numerical value for δ , but we have made no effort to optimize it. A careful estimation would produce a reasonable size for δ .

The cubic fourfold W defined by (1.1) is singular. We note, however, that any singular solution of (1.1) is contained in $W \setminus W^\circ$. We also observe that W° is completely covered by rational planes $x_j = a_j y_j$ ($1 \leq j \leq 3$), which is in sharp contrast to the case of smooth cubic fourfolds.

For a better understanding of the geometry of a singular variety it is necessary to construct a desingularization. In the case of the variety W defined in (1.1), we are in the convenient situation that there exists a *crepant* resolution. Recall that a resolution $f : X \rightarrow W$ of a normal variety with invertible canonical sheaf ω_W is said to be crepant if $f^* \omega_W = \omega_X$ (see [Re1]). Specifically, let $X \subset \mathbb{P}^5 \times \mathbb{P}^2 \times \mathbb{P}^2$ be the tri-projective variety with tri-homogeneous coordinates $(x_1, x_2, x_3, y_1, y_2, y_3; Y_1, Y_2, Y_3; Z_1, Z_2, Z_3)$ defined by the equations

$$\begin{aligned} x_1 Z_1 + x_2 Z_2 + x_3 Z_3 &= 0, \\ y_i Y_j - y_j Y_i &= 0, \quad 1 \leq i < j \leq 3, \\ Y_1 Z_1 &= Y_2 Z_2 = Y_3 Z_3. \end{aligned}$$

In Chapter 3 we shall prove a more general version of the following result.

Theorem 2. *The projection $\mathbb{P}^5 \times \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^5$ restricts to a crepant resolution $f : X \rightarrow W$.*

Manin has put forward a fundamental conjecture relating the geometry of a projective variety to the arithmetic of its rational points ([FMT], [BM]). Originally this conjecture was formulated for smooth Fano varieties. The number of log-powers in an asymptotic formula for the density of rational points is one off the rank of the Picard group, and Peyre [Pe1] (see also [Sa]) has suggested a formula for the leading constant. This has been generalized to large classes of singular Fano varieties by Batyrev and Tschinkel in [BT1]. Since the resolution in Theorem 2 is crepant, X is an “almost Fano” variety in the sense of [Pe2, Def. 3.1] (cf. Lemma 6 below). In particular, Peyre’s “Formule empirique” [Pe2, 5.1] is expected to predict the asymptotic behaviour of the counting function $N(P)$. We will discuss Peyre’s formula in detail in Chapter 5, and we show in Theorem 8 that it agrees with our Theorem 1. In particular, the Tamagawa constant may be interpreted as an adelic volume of the universal torsor over the crepant resolution which we describe in Chapter 4.

Significant progress on the Manin-Peyre conjecture has been made for surfaces. Using a variety of methods from analytic number theory, it has been verified in a number of cases by Browning, de la Brèteche, Derenthal, Peyre and others. The important papers [BaBr, dlB3, BBD, BBP] and the references in [Br] will guide the reader into the realm of the extensive research literature. There are few definitive results on Fano threefolds, and the remarkable paper [dlB4] on the Segre cubic illustrates the additional difficulties that may appear. For cubic fourfolds, we do not know of any asymptotic formulas except for toric varieties related to the hypersurface $x_1 x_2 x_3 = y_1 y_2 y_3$. The investigation of higher-dimensional varieties is an interesting testing ground for more general versions of the Manin-Peyre conjecture, and we hope that the present rather complex example will initiate further research.

Although often quoted as an asymptotic relation, Manin’s conjecture should be considered as a statement concerning the analytic continuation of the corresponding height zeta function. In our case, the height zeta function attached to the non-trivial part of the cubic defined in (1.1) is given by

$$Z(s) = \sum_{(\mathbf{x}, \mathbf{y}) \in W^\circ} H(\mathbf{x}, \mathbf{y})^{-s}, \quad \operatorname{Re} s > 1.$$

A routine partial summation coupled with Theorem 1 yields an analytic continuation for $Z(s)$.

Theorem 3. *Let δ be as in Theorem 1. The height zeta function $Z(s)$ has analytic continuation to a right half plane $\operatorname{Re} s > 1 - \delta$ except for a pole at $s = 1$ of order 5. In the region $\operatorname{Re} s > 1 - \delta$, $|s - 1| > 1/10$ one has the growth estimate $Z(s) \ll |s|$.*

Finally we remark that the variety W carries an algebraic structure. The open subset W° is an abelian group if the product of the two points $(x_1, x_2, x_3, y_1, y_2, y_3)$ and $(x'_1, x'_2, x'_3, y'_1, y'_2, y'_3)$ is defined by

$$(x_1 y'_1 + x'_1 y_1, x_2 y'_2 + x'_2 y_2, x_3 y'_3 + x'_3 y_3, y_1 y'_2, y_2 y'_2, y_3 y'_3).$$

Somewhat more conceptually, let H be the algebraic group of all 2×2 matrices of the form $\begin{pmatrix} b & a \\ 0 & b \end{pmatrix}$ with b invertible. Let $\Psi : H^3 \rightarrow \mathbb{G}_a$ be the homomorphism

$$(1.5) \quad \left(\begin{pmatrix} b_1 & a_1 \\ 0 & b_1 \end{pmatrix}, \begin{pmatrix} b_2 & a_2 \\ 0 & b_2 \end{pmatrix}, \begin{pmatrix} b_3 & a_3 \\ 0 & b_3 \end{pmatrix} \right) \mapsto \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3},$$

and define $G = \ker(\Psi)/\mathbb{G}_m$ where \mathbb{G}_m is embedded into H^3 via $b \mapsto \left(\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \right)$.

Then

$$(1.6) \quad W^\circ \cong G \cong \mathbb{G}_a \times \mathbb{G}_a \times \mathbb{G}_m \times \mathbb{G}_m$$

as groups, and there is a natural open immersion $G \rightarrow W$ and a natural G -action on W . Hence we identify W with the equivariant compactification of G , the product of two additive and two multiplicative groups. In various cases the group structure can be employed to prove an asymptotic formula for the number of points of bounded height using adelic Fourier analysis. This has been carried out for instance for toric varieties in [BT2] (with a different proof in [Sa] and [dlB2]) and equivariant compactifications of additive and certain other groups (e.g. [CLT, STBT]) including a non-commutative example in [TT]. None of these cases cover the situation of a mixed additive and multiplicative abelian group, and the present case seems to be the first example in the literature for a group of the type (1.6). It is possible, however, that an extension of the Fourier analytic techniques could also produce a result similar to the one announced in Theorem 1, and it would be interesting to compare the two approaches.

1.2. The methods. The proof of Theorem 1 draws from a wide range of methods. The main argument that we now describe uses three very different tools, namely elementary lattice point considerations, analytic counting by multiple Mellin integrals, and an Euler product identity for certain multiple Dirichlet series.

The initial step is not new. Rather than counting integral points on the cubic (1.1) directly, we pass in Section 4.1 to a descent variety, a frequently used technique in this context. After a succession of divisibility considerations, one ends up with a bilinear equation. The underlying lattice structure then provides a complete parametrization of the cubic (1.1), see Lemma 8. These elementary arguments can be interpreted in terms of equations for the universal torsor of the variety; we carry this out explicitly in Section 4.2 (cf. the companion Lemma 11) in order to provide further insight into the genesis of the leading term in the asymptotic formula (1.4).

The parametrization now in hand, the count for $N(P)$ has a new interpretation as a 10-dimensional lattice point problem with a strangely shaped boundary. An enveloping argument along the lines of [BlBr] would produce the correct order of magnitude $N(P) \asymp P(\log P)^4$. Alternatively, one may approach the lattice point problem as a heavily convoluted divisor sum, and tackle it by Mellin transform techniques. Due to the complicated boundary conditions, we will require multidimensional Mellin inversion formulae. This part of the argument seems new in this context and should have a wider range of applications to the Manin-Peyre conjecture and cognate problems in diophantine analysis; we shall mention some in Section 1.3 below. It would take us too far afield at this point to comment on the finer structures of our techniques. We content ourselves here with the remark that the method uses a delicate regularization process of a priori divergent integrals and eventually produces an asymptotic formula for a counting function that mimics $N(P)$, but has a smooth weight attached to the variables, see Section 8.1. The proof of Theorem 1 is then completed by removing the weights and computing the main term through the analytic machinery as a certain residue. This approach is inspired by work of de la Bretèche [dlB1], but our situation is rather more involved.

Some of the additional complications are discussed in the final remark of Chapter 7. If we were only interested in the leading term of the asymptotic formula, we could completely dispense with the rather long and technical Chapter 9 (as well as Chapter 6 and Lemma 24) and use a standard Tauberian argument instead. However, the full asymptotic formula and the analytic continuation of the height zeta function in Theorem 3 require the more complex argument.

The Euler product formula (1.4) arises from an identity for a family of multiple Dirichlet series that may be of some independent interest. It can be used successfully in many cases where the analytic method is applicable. Therefore we highlight this auxiliary result as Theorem 5 below, and reserve Chapter 2 for its discussion and demonstration.

The proof of Theorem 2 proceeds in two steps. We first consider the closure $W' \subset \mathbb{P}^5 \times \mathbb{P}^2$ of the graph of the projection $W \setminus \Pi \rightarrow \mathbb{P}^2$ from the plane Π given by $y_1 = y_2 = y_3 = 0$ and show that $\text{pr}_1 : \mathbb{P}^5 \times \mathbb{P}^2 \rightarrow \mathbb{P}^5$ restricts to a crepant birational morphism $g : W' \rightarrow W$. Then, after a base extension of $\text{pr}_2 : W' \rightarrow \mathbb{P}^2$, we obtain a \mathbb{P}^2 -bundle $\lambda : X \rightarrow B$ over a non-singular del Pezzo surface B of degree 6, where X is crepant over W . Since this desingularisation is G -equivariant, we may compute Peyre's alpha invariant in Lemma 5 by means of a result from [TT].

1.3. Further applications. When $y_1 y_2 y_3 \neq 0$, one may rewrite (1.1) as

$$\frac{x_1}{y_1} + \frac{x_2}{y_2} + \frac{x_3}{y_3} = 0,$$

and there is then a natural generalization to more than three summands. When $n > 3$, the solutions of the equation

$$(1.7) \quad \frac{x_1}{y_1} + \frac{x_2}{y_2} + \dots + \frac{x_n}{y_n} = 0$$

are the zeros of a form of degree n in $2n$ variables. A further development of our techniques yields results for this form that are comparable to Theorem 1. In fact, parts of the arguments are carried out for arbitrary n .

From a more arithmetic point of view, one may also count fractional zero sums of bounded height, that is, solutions of (1.7) with $(x_j; y_j) = 1$ and $|x_j|, |y_j| \leq P$. The extra coprimality conditions decrease the power of the logarithm that appears in the asymptotic formula. This phenomenon has been observed for other forms as well, see Fouvry [F] for a discussion in the case $x_0^3 = x_1 x_2 x_3$.

In a different direction, we note that the cubic form on the left of (1.1) is a linear form in \mathbf{x} , and a quadratic form in \mathbf{y} . Therefore (1.1) also defines a singular bi-projective cubic threefold $\tilde{W} \subset \mathbb{P}^2 \times \mathbb{P}^2$, and again one may count its rational points with respect to the anticanonical height and compare the result with the predictions by Manin and Peyre. This amounts to analyzing the number $\tilde{N}(P)$ of non-trivial solutions to (1.1) that satisfy the conditions $(x_1; x_2; x_3) = (y_1; y_2; y_3) = 1$, $x_1 x_2 x_3 y_1 y_2 y_3 \neq 0$ and the size constraints

$$(1.8) \quad 1 \leq |x_i^2 y_j| \leq P \quad (1 \leq i, j \leq 3).$$

Note that \tilde{W} cannot be written as a compactification of a group in a natural way. In [BIBr] we were able to determine the order of magnitude of $\tilde{N}(P)$: with the normalization (1.8) one has

$$P(\log P)^4 \ll \tilde{N}(P) \ll P(\log P)^4.$$

We take the opportunity to relate these estimates to the standard predictions and show that the order of magnitude agrees with the expected one. The singular locus of \tilde{W} is given by the three points $x_i = x_{i+1} = y_i = y_{i+1} = 0$, $i \in \{1, 2, 3\}$, with indices understood modulo 3. Let $\tilde{X} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ be the tri-projective variety with tri-homogeneous coordinates $(\mathbf{x}; \mathbf{y}; \mathbf{z}) = (x_1, x_2, x_3; y_1, y_2, y_3; z_1, z_2, z_3)$ defined by

$$(1.9) \quad x_1 z_1 + x_2 z_2 + x_3 z_3 = 0,$$

$$(1.10) \quad y_1 z_1 = y_2 z_2 = y_3 z_3.$$

Similarly as in Theorem 2 we prove

Theorem 4. *The restriction to \tilde{X} of projection $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$ onto the first two factors is a crepant resolution of \tilde{W} , and one has $\text{rk Pic}(\tilde{X}) = 5$.*

1.4. Leitfaden. We start in Chapter 2 with the graph theoretic proof of Theorem 5 which will be used to compute explicitly the Euler product in (1.4). Chapter 3 features the proof of Theorem 2 (along with Theorem 4). We also take the opportunity to compute Peyre's alpha invariant at the end of this chapter. In Chapter 4 we pass to the universal torsor (see in particular Theorem 7). Then we are prepared to discuss the “empirical formula” suggested by Peyre in Chapter 5. The main ingredient for this is Theorem 8. The remaining chapters are devoted to the analytic proof of Theorem 1. Chapter 6 provides some preliminary upper bounds for $N(P)$ and related quantities that will be needed later. Chapter 7.1 is of technical nature and introduces certain smooth weight functions along with properties of their (multi-dimensional) Mellin transforms; the proofs can safely be skipped at a first reading. Chapter 8 is the heart of the proof. By Mellin transform and contour shifts the asymptotic formula is reduced to the calculation of certain residues. Chapter 9 is again rather technical, and its only purpose is to remove the smooth weights in order to get an asymptotic formula for a count in a box. Finally, the leading coefficient is computed in Chapter 10, completing the proof of Theorem 1.

2. SOME COMBINATORIAL IDENTITIES

2.1. Multiple Dirichlet series with coprimality constraints. For a natural number r let $G = (V, E)$ be any graph on the set of vertices $V = \{1, \dots, r\}$. Then, whenever s_1, \dots, s_r are complex numbers with $\text{Re } s_j > 1$, the series

$$(2.1) \quad D_G(s_1, \dots, s_r) = \sum_{\substack{n_1, \dots, n_r=1 \\ (n_k; n_l)=1 \text{ for } (k,l) \in E}} n_1^{-s_1} n_2^{-s_2} \dots n_r^{-s_r}$$

is absolutely convergent. Note that when $r = 1$, then E is necessarily empty, and (2.1) reduces to the definition of Riemann's zeta function $\zeta(s_1)$. Likewise, for any $r \in \mathbb{N}$, one finds from (2.1) that

$$D_{(V, \emptyset)}(s_1, \dots, s_r) = \zeta(s_1) \dots \zeta(s_r),$$

so that there is an analytic continuation to \mathbb{C}^r except for singularities at $s_j = 1$. The following result describes the situation for any graph $G = (V, E)$. For a subset $U \subset E$ we define its vertex set $\text{ver } U \subset V$ as the set of all vertices that are adjacent to at least one edge in U .

Theorem 5. *Let $\mathbf{s} \in \mathbb{C}^r$ with $\text{Re } s_j > 1$ for $1 \leq j \leq r$. Then*

$$(2.2) \quad \zeta(s_1)^{-1} \dots \zeta(s_r)^{-1} D_G(\mathbf{s}) = \prod_p \sum_{U \subset E} (-1)^{|U|} \prod_{j \in \text{ver } U} p^{-s_j}.$$

The Euler product on the right hand side of (2.2) converges absolutely in the region $\text{Re } s_j > \frac{1}{2}$ ($1 \leq j \leq r$), and constitutes an analytic continuation of the function

$$\Xi_G(\mathbf{s}) = \zeta(s_1)^{-1} \dots \zeta(s_r)^{-1} D_G(\mathbf{s})$$

to this set.

The simplest example not yet considered is when $r = 2$, $E = \{(1, 2)\}$. Here the Euler factors in (2.2) are $1 - p^{-s_1 - s_2}$, and the principal conclusion of Theorem 5 reduces to the familiar identity

$$\sum_{\substack{n_1, n_2=1 \\ (n_1; n_2)=1}}^{\infty} n_1^{-s_1} n_2^{-s_2} = \frac{\zeta(s_1) \zeta(s_2)}{\zeta(s_1 + s_2)}.$$

We also note that it will suffice to establish (2.2), because for any $\emptyset \neq U \subset E$ the vertex set has at least two elements, and consequently, one finds that whenever $\operatorname{Re} s_j > \frac{1}{2}$, then

$$\sum_{j \in \operatorname{ver} U} \operatorname{Re} s_j > 1.$$

Hence the product in (2.2) indeed converges absolutely in the indicated region.

Theorem 5 can be generalized in various ways. For instance, if α_j ($1 \leq j \leq r$) are arbitrary completely multiplicative functions with corresponding Dirichlet series $L_j(s) = \sum_{n=1}^{\infty} \alpha_j(n)n^{-s}$, then the identity

$$(2.3) \quad L_1(s_1)^{-1} \cdots L_r(s_r)^{-1} D_G(\mathbf{s}) = \prod_p \sum_{U \subset E} (-1)^{|U|} \prod_{j \in \operatorname{ver} U} \alpha_j(p) p^{-s_j}$$

holds in the region of absolute convergence. The proof of (2.3) is the same. Martin [Mar, Proposition A.4] has proved a related result for more general multiplicative functions, but only for a complete graph G .

In applications, it is desirable to compute the number $\Xi_G(1, \dots, 1)$ explicitly. By (2.2), one has

$$\Xi_G(1, \dots, 1) = \prod_p \sum_{U \subset E} (-1)^{|U|} p^{-|\operatorname{ver} U|} = \prod_p \sum_{k=0}^r b_k p^{-k}$$

where

$$(2.4) \quad b_k = \sum_{\substack{U \subset E \\ \operatorname{ver} U = k}} (-1)^{|U|}.$$

Note that one has $b_0 = 1, b_1 = 0, b_2 = -|E|$ and

$$(2.5) \quad \sum_{k=0}^r b_k = \sum_{U \subset E} (-1)^{|U|} = 0.$$

In any concrete example, the numbers b_k can be computed via (2.4). In this paper, the case of interest is when $r = 6$ and E is the set of pairs $\{(1, 2), (1, 3), (2, 3), (4, 5), (5, 6), (4, 6), (1, 4), (2, 5), (3, 6)\}$.

$$(2.6) \quad \begin{array}{ccccc} 1 & & & & 4 \\ & \diagdown & & \diagup & \\ & 3 & \text{---} & 6 & \\ & \diagup & & \diagdown & \\ 2 & & & & 5 \end{array}$$

Here one finds that $b_3 = 16, b_4 = -9, b_5 = 0$ and $b_6 = 1$. This gives

$$(2.7) \quad \Xi_G(1, 1, 1, 1, 1, 1) = \prod_p \left(1 - \frac{9}{p^2} + \frac{16}{p^3} - \frac{9}{p^4} + \frac{1}{p^6} \right) = \prod_p \left(1 - \frac{1}{p} \right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2} \right).$$

We briefly indicate how one may compute b_3 . By (2.4), one first has to determine all $U \subset E$ with $|\operatorname{ver} U| = 3$. Such a set U is either a complete subgraph on three vertices, or it consists of two edges with one vertex in common. There are exactly two complete subgraphs on three vertices in G , so this class contributes -2 to b_3 . In order to count the other class of sets U , one first chooses one of the six vertices to determine the vertex that the two edges should have in common. Since each vertex has three adjacent edges, one can then make three choices of two edges. Hence, there are 18 pairs of edges with a vertex in common, and (2.4) yields $b_3 = 18 - 2 = 16$, as required. It is equally straightforward but more elaborate to compute b_4 and b_5 . The coefficient b_6 can then be determined from (2.5). We leave the details to the reader.

2.2. Graphs and power series. In this section we reduce the proof of Theorem 5 to an identity for a power series associated with the graph G . To define this series, let $\delta : \mathbb{N}_0 \rightarrow \{0, 1\}$ be defined by $\delta(0) = 1$, $\delta(n) = 0$ for all $n \geq 1$, and put

$$(2.8) \quad \Delta_G(n_1, \dots, n_r) = \prod_{(k,l) \in E} \delta(n_k n_l).$$

The power series

$$(2.9) \quad T_G(x_1, \dots, x_r) = \sum_{\mathbf{n}=0}^{\infty} \Delta_G(\mathbf{n}) \mathbf{x}^{\mathbf{n}}$$

converges in the disk $|x_j| < 1$ ($1 \leq j \leq r$). It turns out that T_G is a rational function that becomes a polynomial when multiplied with

$$(2.10) \quad \Pi_r(x_1, \dots, x_r) = (1 - x_1)(1 - x_2) \dots (1 - x_r).$$

For any $U \subset E$ let

$$(2.11) \quad \mathbf{x}_U = \prod_{j \in \text{ver } U} x_j.$$

Lemma 1. *One has*

$$\Pi_r(\mathbf{x}) T_G(\mathbf{x}) = \sum_{U \subset E} (-1)^{|U|} \mathbf{x}_U.$$

In the next section, we establish Lemma 1 by induction on r , but now is the time to deduce Theorem 5. The indicator function on the conditions $(n_k; n_l) = 1$ for $(k, l) \in E$ is a multiplicative function on (n_1, \dots, n_r) . Hence, the Dirichlet series (2.1) has an Euler product in its region of absolute convergence. By (2.1), (2.8) and (2.9), this takes the shape

$$D_G(\mathbf{s}) = \prod_p T_G(p^{-s_1}, \dots, p^{-s_r}).$$

Hence, by Lemma 1, (2.10) and the Euler product for Riemann's zeta function,

$$D_G(\mathbf{s}) = \zeta(s_1) \zeta(s_2) \dots \zeta(s_r) \prod_p \sum_{U \subset E} (-1)^{|U|} \prod_{j \in \text{ver } U} p^{-s_j}.$$

This confirms Theorem 5.

2.3. An inductive strategy. It remains to prove Lemma 1. First consider the case where $E = \emptyset$ is the empty set. Then, by (2.8), one has $\Delta_{(V, \emptyset)}(\mathbf{n}) = 1$ for all $\mathbf{n} \in \mathbb{N}^r$, whence

$$T_{(V, \emptyset)}(\mathbf{x}) = \sum_{\mathbf{n}=0}^{\infty} \mathbf{x}^{\mathbf{n}} = (1 - x_1)^{-1} \dots (1 - x_r)^{-1},$$

as is claimed in Lemma 1.

When $r = 1$, then $E = \emptyset$ is the only possibility. When $r = 2$, and E is non-empty, then $E = \{(1, 2)\}$. In this case,

$$T_G(x_1, x_2) = \sum_{n_1, n_2=0}^{\infty} \delta(n_1 n_2) x_1^{n_1} x_2^{n_2} = \sum_{n_1=0}^{\infty} x_1^{n_1} + \sum_{n_2=0}^{\infty} x_2^{n_2} - 1 = \frac{1}{1 - x_1} + \frac{1}{1 - x_2} - 1,$$

which is equivalent to the conclusion of Lemma 1. This settles Lemma 1 when $r = 1$ or 2 . We may now suppose that $r \geq 3$, and that Lemma 1 is already established for smaller values of r . Moreover, we have already dealt with the case where E is empty. In the opposite situation, there is at least one edge in E , and by renumbering the vertices, we may suppose that $(1, 2) \in E$. We then consider the graph $G' = (V, E')$ with $E' = E \setminus \{(1, 2)\}$ and note that (2.8) implies the equation

$$\Delta_G(\mathbf{n}) = \delta(n_1 n_2) \Delta_{G'}(\mathbf{n}).$$

It will now be convenient to write $\mathbf{n} = (n_1, n_2, \mathbf{m})$ with $\mathbf{m} = (n_3, \dots, n_r)$, and likewise, $\mathbf{x} = (x_1, x_2, \mathbf{y})$. Then, since $\delta(n_1 n_2) = 1$ holds if and only if $n_1 n_2 = 0$, one finds from (2.9) that

$$(2.12) \quad \begin{aligned} T_G(\mathbf{x}) &= \sum_{n_2=0}^{\infty} \sum_{\mathbf{m}=0}^{\infty} \Delta_{G'}(0, n_2, \mathbf{m}) x_2^{n_2} \mathbf{y}^{\mathbf{m}} \\ &+ \sum_{n_1=0}^{\infty} \sum_{\mathbf{m}=0}^{\infty} \Delta_{G'}(n_1, 0, \mathbf{m}) x_1^{n_1} \mathbf{y}^{\mathbf{m}} - \sum_{\mathbf{m}=0}^{\infty} \Delta_{G'}(0, 0, \mathbf{m}) \mathbf{y}^{\mathbf{m}}. \end{aligned}$$

Let $G_1 = (V \setminus \{1\}, E_1)$ be the graph that is obtained from G by removing the vertex 1 and all edges $(1, l)$ adjacent to 1. Similarly, let $G_2 = (V \setminus \{2\}, E_2)$ be the graph that is obtained from G by removing the vertex 2 and all edges adjacent to 2. Finally, let $H = (V \setminus \{1, 2\}, E_{1,2})$ be the graph that is the graph G with all edges adjacent to 1 or 2 removed. Then (2.8) implies that $\Delta_{G'}(0, n_2, \mathbf{m}) = \Delta_{G_1}(n_2, \mathbf{m})$, $\Delta_{G'}(n_1, 0, \mathbf{m}) = \Delta_{G_2}(n_1, \mathbf{m})$, and $\Delta_{G'}(0, 0, \mathbf{m}) = \Delta_H(\mathbf{m})$. By (2.9) and (2.12), we infer that

$$T_G(\mathbf{x}) = T_{G_1}(x_2, \mathbf{y}) + T_{G_2}(x_1, \mathbf{y}) - T_H(\mathbf{y}).$$

We may now apply the induction hypothesis three times on the right hand side. For any subgraph $G^* = (V^*, E^*)$ of G we put

$$(2.13) \quad S_{G^*}(\mathbf{x}) = \sum_{U \subset E^*} (-1)^{|U|} \mathbf{x}_U,$$

and then we have by (2.10) that

$$\Pi_r(\mathbf{x}) T_G(\mathbf{x}) = (1 - x_1) S_{G_1}(\mathbf{x}) + (1 - x_2) S_{G_2}(\mathbf{x}) - (1 - x_1)(1 - x_2) S_H(\mathbf{y})$$

where it is worth remarking that S_{G_1} is a function of x_2, \mathbf{y} only, and similarly for S_{G_2} . Now let $Q_j = S_{G_j} - S_H$. Then the previous identity may be rewritten as

$$(2.14) \quad \Pi_r(\mathbf{x}) T_G(\mathbf{x}) = Q_1 + Q_2 + S_H - x_1 Q_1 - x_2 Q_2 - x_1 x_2 S_H.$$

By (2.13), the induction will be complete if the right hand side of (2.14) can be shown to equal S_G . Before proceeding in this direction, we first derive a useful formula for Q_1 . In fact, a set $U \subset E_1 \setminus E_{1,2}$ is characterized by the condition that $2 \in \text{ver } U$. Hence, by (2.13),

$$(2.15) \quad Q_1 = S_{G_1} - S_H = \sum_{\substack{U \subset E_1 \\ 2 \in \text{ver } U}} (-1)^{|U|} \mathbf{x}_U.$$

By symmetry,

$$(2.16) \quad Q_2 = \sum_{\substack{U \subset E_2 \\ 1 \in \text{ver } U}} (-1)^{|U|} \mathbf{x}_U.$$

We now consider S_G and split the sum into various subsums. The subsets $U \subset E$ fall into exactly one of the following eight classes:

$$(I) \ 1 \notin \text{ver } U, 2 \notin \text{ver } U, \quad (II) \ 1 \in \text{ver } U, 2 \notin \text{ver } U, \quad (III) \ 1 \notin \text{ver } U, 2 \in \text{ver } U.$$

Any remaining set $U \subset E$ will have $\{1, 2\} \subset \text{ver } U$. Any set U that contains the edge $(1, 2) \in E$ is of this type, and in this case we write $U = (1, 2) \cup U'$ with $U' = U \setminus \{(1, 2)\}$, and sort these sets into the classes

$$\begin{aligned} (IV) \quad & U = \{(1, 2)\} \cup U'; \quad 1, 2 \notin \text{ver } U', \\ (V) \quad & U = \{(1, 2)\} \cup U'; \quad 1 \in \text{ver } U', \ 2 \notin \text{ver } U', \\ (VI) \quad & U = \{(1, 2)\} \cup U'; \quad 1 \notin \text{ver } U', \ 2 \in \text{ver } U', \\ (VII) \quad & U = \{(1, 2)\} \cup U'; \quad 1, 2 \in \text{ver } U'. \end{aligned}$$

Any set $U \subset E$ that has not been considered must have $1, 2 \in \text{ver } U$, but $(1, 2) \notin U$, and this is the characteristics of sets in the final class (VIII). Accordingly, we sort the sets $U \subset E$ into these classes and obtain the decomposition

$$S_G(\mathbf{x}) = S^{(\text{I})} + S^{(\text{II})} + \dots + S^{(\text{VIII})}$$

where $S^{(\dagger)}$ is the subsum over sets U in class (\dagger) .

It remains to identify the various sums $S^{(\dagger)}$ with certain terms in (2.14). Note that a set $U \subset E$ with $1, 2 \notin \text{ver } U$ is actually a subset of $E_{1,2}$, and *vice versa*. By (2.13), this shows that $S^{(\text{I})} = S_H$, the third summand in (2.14).

Next, consider a subset $U \subset E$ of class (II). The condition that $2 \notin \text{ver } U$ is equivalent with $U \subset E_2$. Then the further requirement that $1 \in \text{ver } U$ in conjunction with (2.16) shows that $S^{(\text{II})} = Q_2$. By symmetry and (2.15), we also have $S^{(\text{III})} = Q_1$. This corresponds to the first two summands in (2.14).

Now consider a set $U = \{(1, 2)\} \cup U'$ of class (V). Then, by (2.11), we have

$$(-1)^{|U|} \mathbf{x}_U = -x_2 (-1)^{|U'|} \mathbf{x}_{U'},$$

and U' is a set of class (II). By summing over U' , the results of the previous paragraph yields $S^{(\text{V})} = -x_2 S^{(\text{II})} = -x_2 Q_2$. By symmetry, one also has $S^{(\text{VI})} = -x_1 Q_1$. The same argument also shows that $S^{(\text{IV})} = -x_1 x_2 S_H$. This identifies the fourth, fifth and last summand on the right hand side of (2.14) as $S^{(\text{VI})}$, $S^{(\text{V})}$ and $S^{(\text{IV})}$, respectively.

Finally, consider a set $U = \{(1, 2)\} \cup U'$ of class (VII). Then U' is of class (VIII). Inversely, when U' is of class (VIII), the set $\{(1, 2)\} \cup U'$ is of class (VII). Since

$$(-1)^{|U|} \mathbf{x}_U = -(-1)^{|U'|} \mathbf{x}_{U'},$$

it follows that $S^{(\text{VII})} = -S^{(\text{VIII})}$. On collecting together, we have now shown that the right hand side of (2.14) equals S_G , completing the induction.

3. RESOLUTION OF SINGULARITIES

In this chapter we prove Theorem 2. In preparation for the verification of Peyre's empirical formula in Chapter 5 we will also compute Peyre's alpha invariant [Pe1, Def. 2.4] for the variety X in Lemma 5 and show in Lemma 6 that X is an "almost Fano" variety in the sense of [Pe2, Def. 3.1].

It amounts to no extra effort to work in more generality and consider varieties and schemes over an arbitrary (fixed) base field k . Let $n \geq 2$ be an arbitrary integer and let $W = W_n \subset \mathbb{P}^{2n-1}$ be the normal projective hypersurface with homogeneous coordinates $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)$ defined by the equation

$$(3.1) \quad x_1 y_2 y_3 \cdots y_{n-1} y_n + x_2 y_1 y_3 \cdots y_n + \cdots + x_n y_1 y_2 \cdots y_{n-1} = 0.$$

As in the special case $n = 3$, $k = \mathbb{Q}$ considered in the introduction, there is a natural G -action on W by a commutative algebraic group G : if $H \subset GL_2$ denotes the subgroup of upper triangular matrices with equal diagonal elements and $\Psi : H^n \rightarrow \mathbb{G}_a$ is the obvious generalization of (1.5), then setting $G = \ker \Psi / \mathbb{G}_m$ we have a natural open immersion

$$j : G \rightarrow W.$$

and a natural action $\alpha : G \times W \rightarrow W$.

The variety W is of multiplicity $n - 1$ along the $(n - 1)$ -plane Π given by $y_1 = y_2 = \dots = y_n = 0$. Let $\Gamma = \text{Bl}_\Pi \mathbb{P}^{2n-1}$ be the blow-up of \mathbb{P}^{2n-1} along Π . It is the subvariety of $\mathbb{P}^{2n-1} \times \mathbb{P}^{n-1}$ with bi-homogeneous coordinates $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n; Y_1, Y_2, \dots, Y_n)$ defined by the equations

$$(3.2) \quad y_i Y_j - y_j Y_i = 0, \quad 1 \leq i < j \leq n.$$

The restriction $p : \Gamma \rightarrow \mathbb{P}^{2n-1}$ of $\mathbb{P}^{2n-1} \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{2n-1}$ to Γ gives a birational morphism, which is an isomorphism outside the exceptional divisor $D = p^{-1}(\Gamma)$ of p . The blow-up $W' = \text{Bl}_\Pi W$ of W along Π is the closure of $p^{-1}(W \setminus \Pi)$ in Γ (see [EH, Prop. IV-21]). Hence W' is the subvariety of Γ defined by the equation

$$(3.3) \quad x_1 Y_2 Y_3 \cdots Y_{n-1} Y_n + x_2 Y_1 Y_3 \cdots Y_n + \dots + x_n Y_1 Y_2 \cdots Y_{n-1} = 0.$$

Let $g : W' \rightarrow W$ be the restriction of p to W' . The inverse image scheme $p^{-1}(W) = W \times_{\mathbb{P}^{2n-1}} \Gamma$ of W is the subscheme of $\mathbb{P}^{2n-1} \times \mathbb{P}^{n-1}$ defined by (3.1). From (3.2) and (3.3) we deduce that

$$(x_1 y_2 y_3 \cdots y_{n-1} y_n + x_2 y_1 \cdots y_{n-1} y_n + \dots + x_n y_1 y_2 \cdots y_{n-1}) Y_i^{n-1} = 0$$

for all $1 \leq i \leq n$. Hence (3.1) holds on W' , and it is therefore a subscheme of $p^{-1}(W)$. As $p^{-1}(W)$ and W' are locally principal subschemes of Γ , we may also regard them as Cartier divisors (see [Ha, p. 145]). By our initial remark that W is of multiplicity $n-1$ along Π , we conclude that $p^{-1}(W) = W' + (n-1)D$ (see [Fu1, Cor. 4.2.2 and Section 4.3]).

We recall that for a normal variety V with open immersion $j : U \rightarrow V$ of the non-singular locus U of V the canonical sheaf ω_V of V is defined by $\omega_V = j_*(\Lambda^{\dim V} \Omega_U)$ where as usual Ω_U denotes the cotangent sheaf (cf. [Re2] and [Ha, pp. 127-128, 180-182] for the notation).

Lemma 2. *The canonical sheaves ω_W and $\omega_{W'}$ are invertible, and there is a canonical isomorphism $g^* \omega_W = \omega_{W'}$ of $O_{W'}$ -modules.*

Proof. The inclusions $i : W \rightarrow \mathbb{P}^{2n-1}$ and $j : W' \rightarrow \Gamma$ embed W and W' as locally principal subschemes of non-singular varieties. Hence by the adjunction formula, there are canonical isomorphisms $i^* \omega_{\mathbb{P}^{2n-1}}(W) = \omega_W$ and $j^* \omega_\Gamma(W') = \omega_{W'}$ induced by the Poincaré residue (cf. Ex. 25 in [Re3, Chapter 3]). This shows the invertibility of ω_W and $\omega_{W'}$ as well as the equality

$$g^* \omega_W = g^* i^* (\omega_{\mathbb{P}^{2n-1}}(W)) = j^* p^* (\omega_{\mathbb{P}^{2n-1}}(W)) = j^* (p^* (\omega_{\mathbb{P}^{2n-1}}) \otimes_{O_\Gamma} O_\Gamma(p^{-1}W)).$$

By [GH, p. 608] and the obvious fact $\dim \mathbb{P}^{2n-1} - \dim \Pi - 1 = n-1$ we have the canonical isomorphism

$$p^* \omega_{\mathbb{P}^{2n-1}} = \omega(-(n-1)D) = \omega_\Gamma \otimes_{O_\Gamma} O_\Gamma(-(n-1)D).$$

Since $O_\Gamma(p^{-1}W) \otimes_{O_\Gamma} O_\Gamma(-(n-1)D) = O_\Gamma(p^{-1}W - (n-1)D)$ (cf. [Ha, p. 144]), we finally obtain canonical isomorphisms

$$g^* \omega_W = j^* (\omega_\Gamma \otimes_{O_\Gamma} O_\Gamma(p^{-1}W - (n-1)D)) = j^* (\omega_\Gamma \otimes_{O_\Gamma} O_\Gamma(W')) = \omega_{W'}.$$

This completes the proof.

Now let $X \subset \mathbb{P}^{2n-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ be the tri-projective variety with tri-homogeneous coordinates $(x_1, \dots, x_n, y_1, \dots, y_n; Y_1, \dots, Y_n; Z_1, \dots, Z_n)$ defined by the equations

$$(3.4) \quad x_1 Z_1 + \dots + x_n Z_n = 0,$$

$$(3.5) \quad y_i Y_j - y_j Y_i = 0, \quad 1 \leq i < j \leq n,$$

$$(3.6) \quad Y_1 Z_1 = \dots = Y_n Z_n.$$

Then (3.6) and (3.4) imply

$$\begin{aligned} & (x_1 Y_2 Y_3 \cdots Y_n + x_2 Y_1 Y_3 \cdots Y_n + \dots + x_n Y_1 \cdots Y_{n-1}) Z_i \\ &= (x_1 Z_1 + \dots + x_n Z_n) Y_1 \cdots Y_{i-1} Y_{i+1} \cdots Y_n = 0. \end{aligned}$$

In particular, (3.3) holds on X , and the projection $\mathbb{P}^{2n-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{2n-1} \times \mathbb{P}^{n-1}$ onto the first two factors restricts to a morphism $h : X \rightarrow W'$. Let $B \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ be the bi-projective variety with bi-homogeneous coordinates $(Y_1, \dots, Y_n; Z_1, \dots, Z_n)$ defined by (3.6). Then we have the following lemma.

Lemma 3. (i) *The variety B is a projective toric variety of dimension $n - 1$, which is locally a complete intersection. Its singular locus is the union of all closed subsets defined by equations*

$$Y_i = Y_j = Z_i = Z_j = 0, \quad 1 \leq i < j \leq n.$$

In particular, if $n \leq 3$, then B is non-singular.

(ii) *The variety X is a \mathbb{P}^{n-1} -bundle over B .*

Proof. (i) Let B° be the open subset of B where none of the Y_i vanishes. By (3.6) this is equivalent to the condition that none of the Z_i vanishes, and we may regard B° as the underlying variety of the $(n - 1)$ -dimensional algebraic torus T obtained as the quotient group of the diagonal embedding of \mathbb{G}_m in \mathbb{G}_m^n . There is a \mathbb{G}_m^n -action on B° given by

$$(t_1, \dots, t_n) \cdot (Y_1, \dots, Y_n; Z_1, \dots, Z_n) = (t_1 Y_1, \dots, t_n Y_n; Z_1/t_1, \dots, Z_n/t_n),$$

and this action factorizes to a T -action $\varrho : T \times B \rightarrow B$ such that its restriction $T \times B^\circ \rightarrow B^\circ$ is the group law on T . Hence B is a toric variety of dimension $n - 1$.

We cover B by open subsets $B_{k,l}$ ($1 \leq k \neq l \leq n$), defined by the conditions $Y_k Z_l \neq 0$. For $i \neq k$ and $j \neq l$ let $t_i = Y_i/Y_k$ and $u_j = Y_j/Y_l$. Using (3.6), we may then eliminate t_l and u_k and identify $B_{k,l}$ with a subvariety of \mathbb{A}^{2n-4} with affine coordinates t_i, u_j for $i, j \in \{1, 2, \dots, n\} \setminus \{k, l\}$, defined by $n - 3$ equations. For instance, if $k = n - 1, l = n$, then $B_{k,l} \subset \mathbb{A}^{2n-4}$ is the complete intersection given by the equations

$$t_1 u_1 - t_2 u_2 = \dots = t_{n-3} u_{n-3} - t_{n-2} u_{n-2} = 0$$

with singular locus given by the union of all closed subsets where $t_i = t_j = u_i = u_j = 0$ for $1 \leq i < j \leq n - 2$. Permuting indices, the same holds for general k, l .

(ii) The projection $\mathbb{P}^{2n-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ restricts to a surjective morphism from X to B where the fibre at a point $(Y_1, \dots, Y_n; Z_1, \dots, Z_n) \in B$ is the $(n - 1)$ -plane in \mathbb{P}^{2n-1} defined by (3.4) and (3.5).

We are now prepared to prove the following more general version of Theorem 2:

Theorem 6. *The projection $\mathbb{P}^{2n-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{2n-1}$ restricts to a proper G -equivariant morphism $f : X \rightarrow W$ from a normal variety X with $f^* \omega_W \cong \omega_X$. If $n = 3$, then f is a crepant resolution of W .*

Proof. Let $h : X \rightarrow W'$ and $g : W' \rightarrow W$ be as above. Then $f = gh$. As we have already shown that g is birational with $g^* \omega_W = \omega_{W'}$, it remains to show that h is a crepant resolution of W' . It follows from Lemma 3 that X is non-singular. Let $U_0 \subset W'$ be the open subset with at most one $Y_j = 0$, and for $1 \leq i \leq n$ let $U_i \subset W'$ be the open subset where $x_i \neq 0$ and $Y_j = 0$ for at most one $j \neq i$. We may then identify local sections $s_i : U_i \rightarrow X$ of h as follows. For a point $P = (x_1, \dots, x_n, y_1, \dots, y_n; Y_1, \dots, Y_n) \in U_i$ we define the coordinates (Z_1, \dots, Z_n) of $s_i(P)$ by

$$(Y_2 Y_3 \cdots Y_n, Y_1 Y_3 \cdots Y_n, \dots, Y_1 Y_2 \cdots Y_{n-1})$$

if $i = 0$, and for $1 \leq i \leq n$ by

$$Z_i = - \sum_{k \neq i} x_k \frac{Y_1 Y_2 \cdots Y_n}{Y_k Y_i}, \quad Z_j = x_i \frac{Y_1 Y_2 \cdots Y_n}{Y_j Y_i}, \quad j \neq i.$$

It is now easy to see that the morphisms glue to a section $s : U = U_0 \cup U_1 \cup \dots \cup U_n \rightarrow X$ of $h : X \rightarrow W'$ and that h maps $h^{-1}(U)$ isomorphically onto U . This shows that h is birational and that the restrictions of $h^* \omega_{W'}$ and ω_X to $h^{-1}(U)$ are isomorphic. Next we show that the restriction from $\text{Pic}(X)$ to $\text{Pic}(h^{-1}(U))$ is bijective. It follows from Lemma 3 that X is locally a complete intersection that is non-singular when $n = 3$, and has singular locus X_{sing} disjoint to $h^{-1}(U)$ and of codimension ≥ 4 if $n \geq 4$. The restriction from $\text{Pic}(X)$ to $\text{Pic}(X \setminus X_{\text{sing}})$ is therefore bijective by a theorem of Grothendieck [Gr1, exp. XI, §3]. Now let $Z = X \setminus h^{-1}(U)$, and for $1 \leq i < j < k \leq n$

let $Z_{i,j,k} \subset Z$ be the subset where $Y_i = Y_j = Y_k = 0$ and for $1 \leq i < j \leq n$ let $Z_{i,j} \subset Z$ be the subset where $x_i = x_j = Y_i = Y_j = 0$. Then Z is the union of all $Z_{i,j,k}$ and all $Z_{i,j}$, and we see that $Z = X \setminus h^{-1}(U)$ is of codimension at least 2 in X . Hence the restriction from $\text{Pic}(X \setminus X_{\text{sing}})$ to $\text{Pic}(h^{-1}(U))$ is also bijective (cf. [Ha, Chapter II, Prop 6.5b and Cor. 6.16]).

It remains to show that the resolution is G -invariant. The set $V = j(G)$ is the open subset of W defined by $y_1 y_2 \cdots y_n \neq 0$. Hence $f^{-1}(V)$ is mapped isomorphically onto V under f , and the inverse map is given by $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (x_1, \dots, x_n, y_1, \dots, y_n; y_1, \dots, y_n; 1/y_1, \dots, 1/y_n)$. We may thus embed G as an open subset of X , and there is a natural G -action $\beta : G \times X \rightarrow X$, given by

$$\left(\begin{pmatrix} b_1 & a_1 \\ 0 & b_1 \end{pmatrix}, \dots, \begin{pmatrix} b_n & a_n \\ 0 & b_n \end{pmatrix} \right) \cdot (x_1, \dots, x_n, y_1, \dots, y_n; Y_1, \dots, Y_n; Z_1, \dots, Z_n) \\ = ((b_1 x_1 + a_1 y_1, \dots, b_n x_n + a_n y_n, b_1 y_1, \dots, b_n y_n; b_1 y_1, \dots, b_n y_n; z_1/b_1, \dots, z_n/b_n).$$

The restriction of β to $G \times f^{-1}j(G)$ reduces to the group law on G , and it is easy to see that there is a commutative square

$$\begin{array}{ccc} G \times X & \xrightarrow{\beta} & X \\ (id, f) \downarrow & & \downarrow f \\ G \times W & \xrightarrow{\alpha} & W \end{array}$$

This completes the proof of the theorem.

The proof of Theorem 4 proceeds along similar lines: We recall the definition of \tilde{W} and \tilde{X} in Section 1.3. Let $\tilde{B} \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the subvariety with bi-homogeneous coordinates $(y_1, y_2, y_3; z_1, z_2, z_3)$ defined by (1.10). It is the blow-up of \mathbb{P}^2 at the three points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and hence a non-singular del Pezzo surface of degree 6 with $\text{Pic}(\tilde{B}) \cong \mathbb{Z}^4$. Moreover, as the map $(\mathbf{x}; \mathbf{y}; \mathbf{z})$ makes \tilde{X} to a \mathbb{P}^1 -bundle over \tilde{B} (cf. (1.9)), we conclude that \tilde{X} is non-singular and $\text{Pic}(\tilde{X}) \cong \mathbb{Z}^5$. We now consider the restriction to \tilde{X} of the projection $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$ which sends $(\mathbf{x}; \mathbf{y}; \mathbf{z})$ to $(\mathbf{x}; \mathbf{y})$. As

$$(x_1 z_1 + x_2 z_2 + x_3 z_3) y_{i+1} y_{i+2} = (x_1 y_2 y_3 + x_2 y_1 y_3 + x_3 y_1 y_2) z_i$$

for $1 \leq i \leq 3$ and with indices modulo 3, we obtain a morphism $\tilde{f} : \tilde{X} \rightarrow \tilde{W}$. If we let $\tilde{X}_0 \subset \tilde{X}$ and $\tilde{W}_0 \subset \tilde{W}$ be the open subsets where $(y_1 y_2, y_1 y_3, y_2 y_3) \neq 0$, the \tilde{f} restricts to an isomorphism $\tilde{f}_0 : \tilde{X}_0 \rightarrow \tilde{W}_0$ with an inverse map $\tilde{g}_0 : \tilde{W}_0 \rightarrow \tilde{X}_0$ which sends $(\mathbf{x}; \mathbf{y})$ to $(\mathbf{x}; \mathbf{y}; \mathbf{z})$ with $z_i = y_{i+1} y_{i+2}$ (again taking indices modulo 3). Hence $\tilde{f} : \tilde{X} \rightarrow \tilde{W}$ is a desingularization of \tilde{W} . Note that \tilde{W} is locally a complete intersection and hence a Gorenstein variety with invertible canonical sheaf $\omega_{\tilde{W}}$. As \tilde{f}_0 is an isomorphism, the canonical homomorphism $f^* \omega_{\tilde{W}} \rightarrow \omega_{\tilde{X}}$ of $O_{\tilde{X}}$ -modules restricts to an isomorphism of $O_{\tilde{X}_0}$ -modules. Since \tilde{X} is smooth and $\text{codim}(\tilde{X} \setminus \tilde{X}_0) \geq 2$, the isomorphism class of an invertible $O_{\tilde{X}}$ -module is uniquely determined by its restriction to \tilde{X}_0 (see [Ha, Chap. II, 6.5, 6.11, 6.15]). So $f^* \omega_{\tilde{W}} \cong \omega_{\tilde{X}}$ as $O_{\tilde{X}}$ -modules, which means that $\tilde{f} : \tilde{X} \rightarrow \tilde{W}$ is crepant.

We return to our original set-up and proceed to compute Peyre's alpha invariant [Pe1, Def 2.4, p. 120] of the non-singular fourfold X in the case $n = 3$. It is convenient to identify G with the open subvariety $f^{-1}(j(G)) \subset X$ where $y_1 y_2 y_3 \neq 0$ and make use of the fact that X is an equivariant compactification of G .

We introduce the following notation: let D_0 be the subvariety of X defined by $y_1 = y_2 = y_3 = 0$. If $1 \leq i \leq 3$ and $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$, we define $D_i \subset X$ by $Y_j = Y_k = 0$ and $D_{i+3} \subset X$ by $Z_j = Z_k = 0$. Then D_0 is a \mathbb{P}^1 -bundle over B contained in $X \setminus G$, while D_1, \dots, D_6 are the inverse images of the exceptional curves on the toric del Pezzo surface B . It follows that $X \setminus G = D_0 \cup D_1 \cup \dots \cup D_6$ and we shall write $\text{Div}_{X \setminus G} X = \sum_{i=0}^6 \mathbb{Z} D_i$ for the free abelian group of divisors with support in

$X \setminus G$. If D is a divisor on X , we will write $[D]$ for its class in $\text{Pic}(X)$ and $C_{\text{eff}}(X) \subset \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ for the pseudo-effective cone spanned by the classes of the effective divisors.

Lemma 4. (i) *The canonical homomorphism from $\text{Div}_{X \setminus G} X$ to $\text{Pic}(X)$ is surjective and its kernel is the subgroup generated by $D_2 - D_1 + D_4 - D_5$ and $D_3 - D_1 + D_4 - D_6$. In particular,*

$$(3.7) \quad \text{rk Pic}(X) = 5.$$

- (ii) *Any element in $C_{\text{eff}}(X)$ is equal to $\sum_{i=0}^6 \lambda_i [D_i]$ for some non-negative real numbers $\lambda_0, \dots, \lambda_7$.*
(iii) *If $1 \leq i \leq 3$ and $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$, then $3(D_0 + D_j + D_k + D_{i+3})$ is an anticanonical divisor.*

Proof. (i) Let $\text{Hom}(G, \mathbb{G}_m)$ be the character group of G . By [TT, Prop. 1.1] there is a natural exact sequence

$$0 \rightarrow \text{Hom}(G, \mathbb{G}_m) \rightarrow \text{Div}_{X \setminus G} X \rightarrow \text{Pic}(X) \rightarrow 0$$

where the map from $\text{Hom}(G, \mathbb{G}_m)$ is the usual divisor map of rational functions. Hence, as $\text{Hom}(G, \mathbb{G}_m)$ is a free abelian group generated by $y_1/y_2 = Y_1/Y_2$ and $y_1/y_3 = Y_1/Y_3$, it suffices to note that $\text{div}(Y_1/Y_2) = D_2 - D_1 + D_4 - D_5$ and $\text{div}(Y_1/Y_3) = D_3 - D_1 + D_4 - D_6$ to get the desired assertion.

(ii) This is a special case of [TT, Prop. 1.1(3)].

(iii) As $f : X \rightarrow W$ is crepant and $\omega_W \cong \mathcal{O}_W(-3)$, it suffices to show that the principal closed subscheme of X defined by y_i gives rise to the divisor $D_0 + D_j + D_k + D_{i+3}$ (cf. [Ha, II.6.17.1]). To see this, we first note that y_i has multiplicity 1 along D_0 . It is therefore enough to prove that the closed subscheme of X defined by Y_i has divisor $D_j + D_k + D_{i+3}$, which is easy to check by computing the divisor of $Y_i = 0$ on B .

Now let $C_{\text{eff}}(X)^\vee \subset \text{Hom}(\text{Pic}(X) \otimes \mathbb{R}, \mathbb{R})$ be the dual cone of all linear maps $\Lambda : \text{Pic}(X) \otimes \mathbb{R} \rightarrow \mathbb{R}$ such that $\Lambda([D]) \geq 0$ for every effective divisor D on X . Moreover, let $l : \text{Hom}(\text{Pic}(X) \otimes \mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$ be the linear map which sends Λ to $\Lambda([-K_X])$. We then endow $\text{Hom}(\text{Pic}(X) \otimes \mathbb{R}, \mathbb{R})$ with the Lebesgue measure ds normalized such that $L = \text{Hom}(\text{Pic}(X), \mathbb{Z})$ has covolume 1, and $H_X = l^{-1}(1)$ with the measure $ds/d(l-1)$. If z_1, \dots, z_r are coordinates for $\text{Hom}(\text{Pic}(X) \otimes \mathbb{R}, \mathbb{R}) = \mathbb{R}^r$ with respect to a \mathbb{Z} -basis of L and $l(z_1, \dots, z_r) = \alpha_1 z_1 + \dots + \alpha_r z_r$, then $ds/d(l-1) = dz_1 \dots \widehat{dz_i} \dots dz_r / |\alpha_i|$ whenever $\alpha_i \neq 0$. After these preparations, we may now define $\alpha(X)$ as

$$(3.8) \quad \alpha(X) = \int_{C_{\text{eff}}(X)^\vee \cap H_X} \frac{ds}{d(l-1)}.$$

The following result evaluates the alpha invariant explicitly.

Lemma 5. *One has*

$$\alpha(X) = \frac{1}{2^4 3^5} = \frac{1}{3888}.$$

Proof. By part (i) of the preceding lemma, the classes of D_0, D_1, D_2, D_3, D_4 form a \mathbb{Z} -basis of $\text{Pic}(X)$. Let e_0, e_1, e_2, e_3, e_4 be the dual \mathbb{Z} -basis of L with $e_i([D_j]) = \delta_{ij}$ and $(z_0, z_1, z_2, z_3, z_4)$ be the coordinates of $\text{Hom}(\text{Pic}(X) \otimes \mathbb{R}, \mathbb{R})$, with respect to this basis. Then, by part (ii) and (iii) of Lemma 4, we have that $C_{\text{eff}}(X)^\vee$ is the subset of $\mathbb{R}_{\geq 0}^5$ defined by the inequalities $z_2 + z_4 - z_1 \geq 0$ and $z_3 + z_4 - z_1 \geq 0$, and that H_X is the hyperplane in \mathbb{R}^5 defined by $3z_0 + 3z_2 + 3z_3 + 3z_4 = 1$. Hence $0 \leq z_0 \leq \frac{1}{3}$ on H_X and

$$\alpha(X) = \int_0^{1/3} \left(\int \int \int \int_{\Delta} \frac{dz_1 dz_2 dz_3 dz_4}{d(l_0 - (1 - 3z_0))} \right) dz_0$$

for $l_0(z_1, \dots, z_4) = 3z_2 + 3z_3 + 3z_4$ and the subset $\Delta \subset \mathbb{R}_{\geq 0}^4$ defined by $z_2 + z_4 - z_1 \geq 0$ and $z_3 + z_4 - z_1 \geq 0$. To compute the inner integral, we substitute $\tilde{z}_i = z_i/(1 - 3z_0)$ for $1 \leq i \leq 4$. Then

$$l_0(\tilde{z}_1, \dots, \tilde{z}_4) - 1 = \frac{l_0(z_1, \dots, z_4) - (1 - 3z_0)}{1 - 3z_0}$$

so that

$$\alpha(X) = \int_0^{1/3} (1 - 3z_0)^3 dz_0 \int \int \int \int_{\Delta} \frac{dz_1 dz_2 dz_3 dz_4}{d(l_0(z_1, \dots, z_4) - 1)}.$$

To compute the last integral, we use the equation $l_0(z_1, \dots, z_4) - 1 = 0$ on H_1 to eliminate z_4 . In this way we see that the multiple integral over Δ equals $\frac{1}{3} \text{Vol}(\Pi)$ for the subset $\Pi \subset \mathbb{R}_{\geq 0}^3$ of the first octant defined by $3z_1 + 3z_2 \leq 1$, $3z_1 + 3z_3 \leq 1$ and $3z_2 + 3z_3 \leq 1$. As $\text{Vol}(\Pi) = \frac{1}{108} = \frac{1}{2^2 3^3}$ and $\int_0^{1/3} (1 - 3z_0)^3 dz_0 = \frac{1}{2^2 3}$, we obtain $\alpha(X) = \frac{1}{2^4 3^5} = \frac{1}{3888}$.

We conclude this chapter by showing that X satisfies the three conditions in [Pe2, Def. 3.1] for being an “almost Fano” variety.

Lemma 6. *Let $X \subset \mathbb{P}^5 \times \mathbb{P}^2 \times \mathbb{P}^2$ be the fourfold defined by (3.4)–(3.6) over some field k . Then the following holds:*

- (i) $H^1(X, O_X) = H^2(X, O_X) = 0$.
- (ii) *The geometric Picard group $\text{Pic}(\bar{k} \times X)$ is torsion-free.*
- (iii) *The anticanonical class is in the interior of $C_{\text{eff}}(X)$.*

Proof. (i) Use Lemma 3(ii) and the fact that $H^1(B, O_B) = H^2(B, O_B) = 0$.
(ii) By Lemma 4(i) we have that $\text{Pic}(K \times X) = \mathbb{Z}^5$ for any field $K \supset k$.
(iii) This follows from Lemma 4(ii) and (iii).

Remark: The variety \tilde{X} featured in the proof of Theorem 4 is also easily seen to be “almost Fano”: as $\tilde{f} : \tilde{X} \rightarrow \tilde{W}$ is a crepant resolution and $\omega_{\tilde{W}}^{-1}$ is ample, $\omega_{\tilde{X}}^{-1}$ is a big $O_{\tilde{X}}$ -module [La, Def. 2.2.1] and hence $[K_{\tilde{X}}]$ in the interior of $C_{\text{eff}}(\tilde{X})$ by [La, Th. 2.2.25]. This proves (iii), while (i) follows as in the previous lemma and (ii) from the fact that \tilde{X} is a \mathbb{P}^1 -bundle over \tilde{B} .

4. THE DESCENT VARIETY

In this chapter, we show that the cubic can be parametrized. We start with simple divisibility considerations that resemble the argument in [BIBr]. In the following section we then show that the descent variety so obtained is the universal torsor.

4.1. An elementary argument. Let \mathcal{W} denote the set of integer solutions to (1.1) and (1.3) with no further coprimality conditions. As a first preparatory step we will link this with the bilinear equation

$$(4.1) \quad u_1 v_1 + u_2 v_2 + u_3 v_3 = 0.$$

For $(\mathbf{x}, \mathbf{y}) \in \mathcal{W}$ we put

$$u = (y_1; y_2; y_3), \quad u_1 = (y_2/u; y_3/u), \quad u_2 = (y_3/u; y_1/u), \quad u_3 = (y_1/u; y_2/u)$$

and observe that

$$(4.2) \quad (u_1; u_2) = (u_2; u_3) = (u_3; u_1) = 1.$$

Hence we can write

$$(4.3) \quad y_1 = uu_2 u_3 w_1, \quad y_2 = uu_1 u_3 w_2, \quad y_3 = uu_1 u_2 w_3$$

with integers $w_j \neq 0$, and the equation (1.1) now reads

$$(4.4) \quad u_1 x_1 w_2 w_3 + u_2 x_2 w_1 w_3 + u_3 x_3 w_1 w_2 = 0.$$

By construction, the coprimality conditions

$$(4.5) \quad (u_j; w_j) = (w_1; w_2) = (w_2; w_3) = (w_3; w_1) = 1 \quad (1 \leq j \leq 3)$$

hold in addition to (4.2). By (4.4), we see that $w_1 \mid u_1 x_1 w_2 w_3$, and (4.5) then implies that $w_1 \mid x_1$. By symmetry, it also follows that $w_2 \mid x_2$, $w_3 \mid x_3$, and we write

$$(4.6) \quad x_j = w_j v_j$$

with $v_j \in \mathbb{Z}$. In this notation, (4.4) reduces to (4.1). These transformations can be reversed: if natural numbers u, u_1, u_2, u_3 and integers v_j, w_j are given, then the numbers x_j, y_j defined by (4.6) and (4.3) satisfy (1.1). In particular, this proves the following.

Lemma 7. *Let \mathcal{A} denote the set of all 10-tuples $u, u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3$ with $u, u_j \in \mathbb{N}$, $w_j \in \mathbb{Z} \setminus \{0\}$, $v_j \in \mathbb{Z}$ that satisfy (4.1), (4.2) and (4.5). Then the map $\mathcal{A} \rightarrow \mathbb{Z}^6$ defined by (4.3) and (4.6) is a bijection between \mathcal{A} and \mathcal{W} .*

For the next step, consider $\mathbf{u} \in \mathbb{N}^3$ as fixed, and study the set $\mathcal{L}(\mathbf{u})$ of solutions $\mathbf{v} \in \mathbb{Z}^3$ of (4.1) as a lattice. For any integers r_1, r_2, r_3 , the numbers

$$(4.7) \quad v_1 = u_2 r_3 - u_3 r_2, \quad v_2 = u_3 r_1 - u_1 r_3, \quad v_3 = u_1 r_2 - u_2 r_1$$

are a solution of (4.1). Fix a complete set \mathcal{S} of residues modulo u_1 , and consider (4.7) as a map

$$\mathcal{S} \times \mathbb{Z} \times \mathbb{Z} \rightarrow \mathcal{L}(\mathbf{u}), \quad \mathbf{r} \mapsto \mathbf{v}.$$

If (4.2) holds, this map is actually a bijection. This fact is certainly well known, but we include the simple proof for completeness: Suppose that \mathbf{r}, \mathbf{r}' with $r_1, r'_1 \in \mathcal{S}$ map to the same $\mathbf{v} \in \mathcal{L}(\mathbf{u})$. Then by (4.7) for v_3 , one finds that $u_2 r_1 \equiv u_2 r'_1 \pmod{u_1}$, and hence that $r_1 = r'_1$. By (4.7) again, it is now immediate that $\mathbf{r} = \mathbf{r}'$, as required to show that the map is injective. To show that the map is surjective, let $\mathbf{v} \in \mathcal{L}(\mathbf{u})$. By (4.2), there are integers a, b with $v_1 = u_2 a - u_3 b$. Then, for any $k \in \mathbb{Z}$, one has

$$v_1 = u_2(a + ku_3) - u_3(b + ku_2).$$

Similarly, there are integers r_1, r_3 with $v_2 = u_3 r_1 - u_1 r_3$. Injecting these expressions into (4.1), we deduce that $u_1 u_2(a - r_3) \equiv u_1 v_1 + u_2 v_2 \equiv 0 \pmod{u_3}$, and hence we may choose k such that $r_3 = a + ku_3$. With this choice, we put $r_2 = b + ku_2$. Then, by construction, the first two equations in (4.7) hold. The third equation must then also hold, because \mathbf{r} maps to the solution of (4.1) with given values v_1, v_2 . This shows that any solution of (4.1) can be written as in (4.7), for some $\mathbf{r} \in \mathbb{Z}^3$. For any $j \in \mathbb{Z}$, the transformation $(r_1, r_2, r_3) \mapsto (r_1 + ju_1, r_2 + ju_2, r_3 + ju_3)$ leaves (4.7) invariant. Hence, an appropriate choice of j guarantees that $r_1 \in \mathcal{S}$, as required. This last invariance property also shows that whenever r_1, r'_1 are natural numbers with $r_1 \equiv r'_1 \pmod{u_1}$, then the sets $\mathcal{R}(r_1) = \{(r_1, r_2, r_3) : r_2, r_3 \in \mathbb{Z}\}$ and $\mathcal{R}(r'_1)$ are mapped to the same image.

We may now use (4.7) within the conclusion of Lemma 7. This yields the following.

Lemma 8. *Let $\mathcal{S}(q)$ denote a complete set of residues modulo q . Let \mathcal{B} denote the set of all 10-tuples $u, u_1, u_2, u_3, w_1, w_2, w_3, r_1, r_2, r_3$ with $u, u_j \in \mathbb{N}$, $w_j \in \mathbb{Z} \setminus \{0\}$, $r_1 \in \mathcal{S}(d_1)$, $r_2 \in \mathbb{Z}$, $r_3 \in \mathbb{Z}$ that satisfy (4.2) and (4.5). Then the map $\mathcal{B} \rightarrow \mathbb{Z}^6$ defined by (4.3) and*

$$x_1 = w_1(u_2 r_3 - u_3 r_2), \quad x_2 = w_2(u_3 r_1 - u_1 r_3), \quad x_3 = w_3(u_1 r_2 - u_2 r_1)$$

is a bijection between \mathcal{B} and \mathcal{W} .

It will be relevant later to know that products $r_i u_j w_k$ with $\{i, j, k\} = \{1, 2, 3\}$ are not much larger than the original variables x_j, y_j . The following lemma makes this precise.

Lemma 9. *Let $(\mathbf{x}, \mathbf{y}) \in \mathcal{W}$ with $|x_j| \leq P$, $|y_j| \leq P$ for $1 \leq j \leq 3$. Suppose that $\mathcal{S}(u_1) \subset [1, 2u_1]$ and*

$$(4.8) \quad |w_1 u_2 u_3| \leq 2 \min(|w_2 u_1 u_3|, |w_3 u_1 u_2|).$$

Then one has

$$(4.9) \quad |r_1 u_2 w_3| \leq 2P, \quad |r_1 u_3 w_2| \leq 2P, \quad |r_2 u_1 w_3| \leq 3P, \quad |r_3 u_1 w_2| \leq 3P,$$

$$(4.10) \quad |r_2 u_3 w_1| \leq 7P, \quad |r_3 u_2 w_1| \leq 7P.$$

Proof. By Lemma 8 and (4.3), the conditions $|x_j|, |y_j| \leq P$ may be rewritten as the six constraints

$$(4.11) \quad uu_1u_2w_3 \leq P, \quad uu_2u_3w_1 \leq P, \quad uu_1u_3w_2 \leq P,$$

and

$$(4.12) \quad |u_2r_3 - u_3r_2| \leq \frac{P}{|w_1|}, \quad |u_3r_1 - u_1r_3| \leq \frac{P}{|w_2|}, \quad |u_1r_2 - u_2r_1| \leq \frac{P}{|w_3|}.$$

Using the bound $1 \leq r_1 \leq 2u_1$, one notes that (4.11) implies the first two bounds in (4.9). These together with (4.12) imply the rest of (4.9). With (4.9) in hand, one applies (4.8) and bounds the minimum by the geometric mean to conclude that

$$\min(|r_3u_2w_1|, |r_2u_3w_1|) \leq \sqrt{|r_2r_3u_2u_3w_1^2|} \leq \frac{3P|w_1u_2u_3|}{\sqrt{|w_2u_3u_1w_3u_2u_1|}} \leq 6P.$$

Appealing to (4.12) once again, we derive (4.10).

4.2. The universal torsor. In Lemma 7 we proved a useful parametrization of the cubic (1.1) in an elementary *ad hoc* fashion. In this section we take a very different route, and use much more sophisticated tools, to compute the universal torsor of the variety (1.1) by applying the general theory of Colliot-Thélène and Sansuc [CS]. The main result of this section is Theorem 7. As a corollary we obtain a new proof of Lemma 7 which is contained in the equivalent companion Lemma 11 below.

Let K be a perfect field with algebraic closure \bar{K} , $\mathfrak{g} = \text{Gal}(\bar{K}/K)$ and $\bar{V} = \bar{K} \times_K V$ for a variety V over K . We recall that an X -torsor over a K -variety X under a K -torus T is a principal homogeneous space $\mathcal{T} \rightarrow X$ under T (see [Mi, Ch. III, §4]). The isomorphism classes $[\mathcal{T}]$ of X -torsors under T are in bijection with elements of $H_{\text{et}}^1(X, T)$ (cf. [CS, Section 1.2]), and if X is a smooth, geometrically integral K -variety with $H_{\text{et}}^0(X, \mathbb{G}_m) = K^*$, then there is a natural exact sequence [CS, 2.0.2]

$$(4.13) \quad 0 \longrightarrow H_{\text{et}}^1(K, T) \longrightarrow H_{\text{et}}^1(X, T) \xrightarrow{\chi} \text{Hom}_{\mathfrak{g}}(\hat{T}, \text{Pic}(\bar{X}))$$

where $\chi([\mathcal{T}]) \in \text{Hom}_{\mathfrak{g}}(\hat{T}, \text{Pic}(\bar{X}))$ sends a character $\Psi : \bar{T} \rightarrow \mathbb{G}_{m, \bar{K}}$ to the \bar{X} -torsor $\bar{\mathcal{T}} \times^{\bar{T}} \mathbb{G}_{m, \bar{K}}$ under $\mathbb{G}_{m, \bar{K}}$, which one obtains from $\bar{\mathcal{T}}$ and Ψ by changing the structure group of the torsor from \bar{T} to $\mathbb{G}_{m, \bar{K}}$. The image $\chi([\mathcal{T}]) \in \text{Hom}_{\mathfrak{g}}(\hat{T}, \text{Pic}(\bar{X}))$ is called the *type* of the torsor. If the K -torus T is split, then $H_{\text{et}}^1(K, T) = 0$ by Hilbert's Theorem 90 (cf. [Mi, III.4.9]). As all K -tori in this paper are split, we shall therefore (by a slight abuse of language) refer to *the* X -torsor of a certain type.

Now suppose that $\text{Pic}(\bar{X})$ is finitely generated and torsion free and that T is the dual torus with character group $\hat{T} = \text{Pic}(\bar{X})$ (here we identify canonically isomorphic \mathfrak{g} -modules). Then an X -torsor \mathcal{T} under T is said to be *universal* if $\chi([\mathcal{T}]) : \hat{T} \rightarrow \text{Pic}(\bar{X})$ is the identity map. It is known [CS, 2.2.9] that a universal torsor exists whenever $X(K) \neq \emptyset$.

We specialize now to the situation relevant in our case. Let $K = \mathbb{Q}$ and $X \subset \mathbb{P}^5 \times \mathbb{P}^2 \times \mathbb{P}^2$ be the fourfold given by (3.4) – (3.6). It is a hypersurface in the fivefold $\Xi \subset \mathbb{P}^5 \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by (3.5) and (3.6). The projection $\mathbb{P}^5 \times \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$ restricts to morphisms $\lambda : X \rightarrow B$ and $\gamma : \Xi \rightarrow B$, which makes X a \mathbb{P}^2 -bundle and Ξ a \mathbb{P}^3 -bundle over the surface $B \subset \mathbb{P}^2 \times \mathbb{P}^2$ defined by (3.6).

As a first step, we will describe the universal torsor over Ξ , which is a (split) smooth projective toric variety: the torus is the open subset $U \subset \Xi$ where all coordinates are different from zero and the U -action $U \times \Xi \rightarrow \Xi$ is given by coordinate-wise multiplication of the two 12-tuples representing points in U and Ξ . It was shown in [Sa, Prop. 8.5] that the universal torsor \mathcal{T} of a split smooth projective toric variety Ξ coincides with the toric morphism from the open toric subvariety $\mathbb{A}^n \setminus F$ of \mathbb{A}^n described by Cox in [Co]. Here the n affine coordinates t_ρ of $\mathcal{T} \subset \mathbb{A}^n$ are indexed by the one-dimensional cones (or edges) of the fan Δ of Ξ (see [Fu2]) and $F \subset \mathbb{A}^n$ is the closed subset defined by the monomials $t^\sigma = \prod_{\rho \notin \sigma(1)} t_\rho$ for the maximal cones σ of Δ .

Lemma 10. *Let $\Omega \subset \mathbb{A}^{10}$ be the open subvariety with coordinates $(\xi_0, \xi_1, \xi_2, \xi_3, u_1, u_2, u_3, w_1, w_2, w_3)$ defined by*

$$(4.14) \quad \begin{aligned} &u_i u_k w_j w_k \neq 0 \text{ for at least one triple } \{i, j, k\} = \{1, 2, 3\}, \\ &(\xi_0, \xi_1, \xi_2, \xi_3) \neq (0, 0, 0, 0). \end{aligned}$$

Let $\varphi : \Omega \rightarrow \Xi$ be the morphism which sends $(\xi_0, \xi_1, \xi_2, \xi_3, u_1, u_2, u_3, w_1, w_2, w_3)$ to

$$(4.15) \quad \begin{aligned} (x_1, x_2, x_3, y_1, y_2, y_3) &= (\xi_1, \xi_2, \xi_3, \xi_0 u_2 u_3 w_1, \xi_0 u_1 u_3 w_2, \xi_0 u_1 u_2 w_3), \\ (Y_1, Y_2, Y_3; Z_1, Z_2, Z_3) &= (u_2 u_3 w_1, u_1 u_3 w_2, u_1 u_2 w_3; u_1 w_2 w_3, u_2 w_1 w_3, u_3 w_1 w_2). \end{aligned}$$

Then $\varphi : \Omega \rightarrow \Xi$ is the underlying morphism of a universal torsor over Ξ .

Proof. For a cone τ of Δ under the action of the torus U of Ξ let $V(\tau)$ be the closure of the orbit O_τ , see [Fu2, Section 3.1]. There is a bijection between edges $\varrho \in \Delta$ and irreducible components $D_\varrho = V(\varrho)$ of $\Xi \setminus U$, and there is also a bijection between maximal cones $\sigma \in \Delta$ and fixed points $P_\sigma = V(\sigma) = O_\sigma \in \Xi \setminus U$ under the action of U . Moreover, we have $\varrho \in \sigma(1)$ if and only if $P_\sigma \in D_\varrho$. There are ten irreducible components D_ϱ of $\Xi \setminus U$. For $i = 1, 2, 3$ we let ξ_i correspond to the prime divisor where $x_i = 0$ and ξ_0 to the prime divisor where $y_1 = y_2 = y_3 = 0$. For a triple $\{i, j, k\} = \{1, 2, 3\}$ we let u_j be the coordinate corresponding to the prime divisor where $Y_i = Y_k = Z_j = 0$ and w_k the coordinate corresponding to the prime divisor where $Y_k = Z_i = Z_j = 0$. There is then a natural embedding of the universal Ξ -torsor \mathcal{T} in the affine space \mathbb{A}^{10} with coordinates $(\xi_0, \xi_1, \xi_2, \xi_3, u_1, u_2, u_3, w_1, w_2, w_3)$.

A point on $\Xi \subset \mathbb{P}^5 \times \mathbb{P}^2 \times \mathbb{P}^2$ is a fixed point P_σ under U if and only if its image has exactly one non-zero coordinate under each of the projections $\text{pr}_1 : \Xi \rightarrow \mathbb{P}^5$, $\text{pr}_2 : \Xi \rightarrow \mathbb{P}^2$ and $\text{pr}_3 : \Xi \rightarrow \mathbb{P}^2$. Such a point either satisfies $y_j Y_j Z_i(P_\sigma) \neq 0$ or $z_l Y_j Z_i(P_\sigma) \neq 0$ for $i \neq j$ and $1 \leq l \leq 3$. In the first case, P_σ does not lie on the divisors corresponding to $\xi_0, u_i, u_k, w_j, w_k$ and in the second case, P_σ does not lie on the divisors corresponding to $\xi_l, u_i, u_k, w_j, w_k$. The exceptional set $F \subset \mathbb{A}^{10}$ of Cox is thus given by the monomials $\xi_l u_i u_k w_j w_k$ where (i, j, k, l) runs over all quadruples with $\{i, j, k\} = \{1, 2, 3\}$ and $0 \leq l \leq 3$. Hence $\mathcal{T} = \mathbb{A}^{10} \setminus F$ is just the open subset $\Omega \subset \mathbb{A}^{10}$ defined in (4.14).

The structure morphism $\varphi : \mathcal{T} \rightarrow \Xi$ of the universal torsor is given in terms of fans in [Co] and [Sa, Prop. 8.5]; it follows from this or from the general local description of torsors [CS, Section 2.3] that the restriction of φ to $\mathbb{G}_{m, \mathbb{Q}}^{10}$ is the homomorphism of tori $\mathbb{G}_{m, \mathbb{Q}}^{10} \rightarrow U$ dual to the divisor map $\mathbb{Q}[U]^* / \mathbb{Q}^* \rightarrow \text{Div}_{\Xi \setminus U}(\Xi)$, where the latter denotes the free group of divisors in Ξ with support in $\Xi \setminus U$. A set of generators of $\mathbb{Q}[U]^* / \mathbb{Q}^*$ is given by $x_1/y_3, x_2/y_3, x_3/y_3, y_1/y_3$ and y_2/y_3 . Computing the divisors of this set, we can determine the restriction of φ to the open subset of Ω where all coordinates are different from zero. We conclude that it is given by (4.15) on this Zariski dense subset and hence everywhere on Ω .

Having completed the proof of Lemma 10, we proceed to relate this result to the universal torsor over X . By the Leray spectral sequence $H_{\text{et}}^p(\bar{B}, R^q \bar{\lambda}_* \mathbb{G}_m) \rightarrow H_{\text{et}}^{p+q}(\bar{X}, \mathbb{G}_m)$ (see [Gr2, (4.5)]) applied to the corresponding morphisms $\bar{\lambda} : \bar{X} \rightarrow \bar{B}$, $\bar{\gamma} : \bar{\Xi} \rightarrow \bar{B}$ over $\bar{\mathbb{Q}}$, there is a commutative diagram with exact rows of trivial \mathfrak{g} -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Pic}(\bar{B}) & \xrightarrow{\bar{\gamma}^*} & \text{Pic}(\bar{\Xi}) & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\ & & \text{id} \downarrow & & \downarrow & & \downarrow \text{id} \\ 0 & \longrightarrow & \text{Pic}(\bar{B}) & \xrightarrow{\bar{\lambda}^*} & \text{Pic}(\bar{X}) & \longrightarrow & \mathbb{Z} \longrightarrow 0 \end{array}$$

where $\bar{\gamma}^*$ and $\bar{\lambda}^*$ are the contravariant functorial maps from $H_{\text{et}}^1(\bar{B}, \mathbb{G}_m) = \text{Pic}(\bar{B})$. The restriction $\text{Pic}(\bar{\Xi}) \rightarrow \text{Pic}(\bar{X})$ is thus an isomorphism, and there is a dual sequence of \mathbb{Q} -tori

$$1 \rightarrow \mathbb{G}_m \rightarrow T \rightarrow S \rightarrow 1$$

where the character groups of T and S are given by $\widehat{T} = \text{Pic}(\bar{\Xi}) = \text{Pic}(\bar{X})$ and $\widehat{S} = \text{Pic}(\bar{B})$. From the functoriality of (4.13) under $X \rightarrow \Xi$ we conclude that the universal Ξ -torsor under T restricts to the universal X -torsor under T . We now restrict the map φ defined in Lemma 10 to the closed subset $\varphi^{-1}(X) \subset \Omega$ defined by (3.4). If we apply (4.15), then (3.4) takes the form

$$(4.16) \quad \xi_1 u_1 w_2 w_3 + \xi_2 u_2 w_1 w_3 + \xi_3 u_3 w_1 w_2 = 0.$$

We may now define three regular functions v_1, v_2, v_3 on $\varphi^{-1}(X)$ as follows. For notational simplicity we agree that all indices are understood modulo 3. On the principal open subset where $u_i w_{i+1} w_{i+2} \neq 0$, we let

$$v_i = -\frac{u_{i+1} w_{i+2} \xi_{i+1} + u_{i+2} w_{i+1} \xi_{i+2}}{u_i w_{i+1} w_{i+2}}, \quad v_{i+1} = \frac{\xi_{i+1}}{w_{i+1}}, \quad v_{i+2} = \frac{\xi_{i+2}}{w_{i+2}}.$$

If in addition $w_i \neq 0$, then $v_i = \xi_i/w_i$ by (4.16), such that v_1, v_2, v_3 are well-defined. If we write u instead of ξ_0 , we obtain the following main result of this section.

Theorem 7. *Let $O \subset \mathbb{A}^{10}$ be the subvariety with coordinates $(u, v_1, v_2, v_3, u_1, u_2, u_3, w_1, w_2, w_3)$ defined by (4.1) and*

$$(4.17) \quad u_i u_k w_j w_k \neq 0 \text{ for at least one triple } \{i, j, k\} = \{1, 2, 3\},$$

$$(4.18) \quad (u, v_1, v_2, v_3) \neq (0, 0, 0, 0).$$

Let $\varphi_O : O \rightarrow X$ be the morphism which sends $(u, v_1, v_2, v_3, u_1, u_2, u_3, w_1, w_2, w_3)$ to

$$(4.19) \quad \begin{aligned} (x_1, x_2, x_3, y_1, y_2, y_3) &= (v_1 w_1, v_2 w_2, v_3 w_3, u u_2 u_3 w_1, u u_1 u_3 w_2, u u_1 u_2 w_3), \\ (Y_1, Y_2, Y_3; Z_1, Z_2, Z_3) &= (u_2 u_3 w_1, u_1 u_3 w_2, u_1 u_2 w_3; u_1 w_2 w_3, u_2 w_1 w_3, u_3 w_1 w_2). \end{aligned}$$

Then $\varphi_O : O \rightarrow X$ is the underlying X -scheme of a universal torsor over X .

Indeed, it follows easily from the definition of the v_i that (4.16) and (4.1) as well as the second condition in (4.14) and (4.18) are equivalent if (4.17) holds.

Finally we turn our attention to integral points. Let $\underline{X} \subset \mathbb{P}_{\mathbb{Z}}^5 \times \mathbb{P}_{\mathbb{Z}}^2 \times \mathbb{P}_{\mathbb{Z}}^2$ be defined by (3.4) – (3.6) and let $\underline{\Xi} \subset \mathbb{P}_{\mathbb{Z}}^5 \times \mathbb{P}_{\mathbb{Z}}^2 \times \mathbb{P}_{\mathbb{Z}}^2$ be defined by (3.5) and (3.6). We may extend the Cox morphism $\varphi : \Omega \rightarrow \Xi$ from Lemma 10 to a morphism $\varphi : \underline{\Omega} \rightarrow \underline{\Xi}$ between toric schemes, since the Cox morphism is derived from a morphism of fans (see [Fu2, pp. 22–23]). By repeating the arguments in the proof of Lemma 10 over \mathbb{Z} , one obtains an open subscheme $\underline{\Omega}$ of $\mathbb{A}_{\mathbb{Z}}^{10}$ with coordinates $(\xi_0, \xi_1, \xi_2, \xi_3, u_1, u_2, u_3, w_1, w_2, w_3)$ defined by (4.14). The morphism $\varphi : \underline{\Omega} \rightarrow \underline{\Xi}$ defined by (4.15) is the underlying morphism of a torsor $\varphi_{\underline{T}} : \underline{T} \rightarrow \underline{\Xi}$ under a split \mathbb{Z} -torus $\underline{T} \cong \mathbb{G}_{m, \mathbb{Z}}^5$ with $H_{\text{et}}^1(\mathbb{Z}, \underline{T}) = 1$ (cf. [Mi, III.4.9]). The \mathbb{Z} -torsor obtained by base extension of $\varphi_{\underline{T}} : \underline{T} \rightarrow \underline{\Xi}$ to an integral point is therefore always trivial. Hence there is a bijection between $\underline{T}(\mathbb{Z})$ -orbits of integral points on $\underline{\Omega}$ and integral points on $\underline{\Xi}$.

If we restrict φ to the closed subset $\varphi^{-1}(X)$ of $\underline{\Omega}$ defined by (4.19), we may again introduce new coordinates such that $\underline{O} = \varphi^{-1}(X)$ is the (locally closed) subscheme of $\mathbb{A}_{\mathbb{Z}}^{10}$ defined by (4.1), (4.17), (4.18), and $\varphi_{\underline{O}} : \underline{O} \rightarrow \underline{X}$ is given by (4.19).

We are now ready to state and prove the following equivalent version of Lemma 7.

Lemma 11. *Let \mathcal{A}_0 denote the set of 10-tuples $(u, v_1, v_2, v_3, u_1, u_2, u_3, w_1, w_2, w_3)$ with $v_j \in \mathbb{Z}$, $u, u_j \in \mathbb{N}$ and $w_j \in \mathbb{Z} \setminus \{0\}$ satisfying (4.1) as well as the coprimality conditions*

$$(4.20) \quad (u_1 u_2 w_1 w_3; u_1 u_2 w_2 w_3; u_1 u_3 w_1 w_2; u_1 u_3 w_2 w_3; u_2 u_3 w_1 w_2; u_2 u_3 w_1 w_3) = 1,$$

$$(4.21) \quad (u; v_1 w_1; v_2 w_2; v_3 w_3) = 1.$$

Then the map $\mathcal{A}_0 \rightarrow \mathbb{Z}^6$ defined by (4.3) and (4.6) gives a bijection between \mathcal{A}_0 and the set of primitive integral solutions to (1.1) and (1.3).

Note that (4.20) is equivalent to (4.2) and (4.5), in which case $(u_2u_3w_1; u_1u_3w_2; u_1u_2w_3) = 1$, so that (4.21) is equivalent to $(x_1; x_2; x_3; y_1; y_2; y_3) = 1$.

Proof. Let $\underline{W} \subset \mathbb{P}_{\mathbb{Z}}^5$ be the subscheme defined by (1.1) and $\underline{f} : \underline{X} \rightarrow \underline{W}$ be the extension of the resolution $f : X \rightarrow W$ induced by the projection $\mathbb{P}_{\mathbb{Z}}^5 \times \mathbb{P}_{\mathbb{Z}}^2 \times \mathbb{P}_{\mathbb{Z}}^2 \rightarrow \mathbb{P}_{\mathbb{Z}}^5$. Then there are natural bijections $\underline{X}(\mathbb{Z}) = X(\mathbb{Q})$, $\underline{W}(\mathbb{Z}) = W(\mathbb{Q})$ and $X^\circ(\mathbb{Q}) = W^\circ(\mathbb{Q})$ for the open subsets $X^\circ \subset X$, $W^\circ \subset W$ where $y = y_1y_2y_3 \neq 0$. If we let $\underline{X}(\mathbb{Z})^\circ \subset \underline{X}(\mathbb{Z})$ correspond to $X^\circ(\mathbb{Q}) \subset X(\mathbb{Q})$ and $\underline{W}(\mathbb{Z})^\circ \subset \underline{W}(\mathbb{Z})$ correspond to $W^\circ(\mathbb{Q}) \subset W(\mathbb{Q})$, then we get a bijection $\underline{X}(\mathbb{Z})^\circ = \underline{W}(\mathbb{Z})^\circ$. Next, let $O^\circ \subset O$ be the open subset where $y_1y_2y_3 = u^3(u_1u_2u_3)^2w_1w_2w_3 \neq 0$ and let $\underline{O}(\mathbb{Z})^\circ \subset \underline{O}(\mathbb{Z})$ correspond to $O^\circ(\mathbb{Q}) \subset O(\mathbb{Q})$ under the bijection $\underline{O}(\mathbb{Z}) = O(\mathbb{Q})$. As $\varphi^{-1}(W^\circ) = O^\circ$, we obtain a bijection between the $\underline{T}(\mathbb{Z})$ -orbits in $\underline{O}(\mathbb{Z})^\circ$ and the points in $\underline{X}(\mathbb{Z})^\circ$. We observe that an integral 10-tuple $(u, v_1, v_2, v_3, u_1, u_2, u_3, w_1, w_2, w_3)$ belongs to $\underline{O}(\mathbb{Z})$ if and only if (4.1), (4.17) and (4.18) hold for all reductions modulo p . Hence it is in $\underline{O}(\mathbb{Z})$ if and only if (4.1), (4.20) and (4.21) hold. There are $2^{\dim T} = 32$ integral points in each $\underline{T}(\mathbb{Z})$ -orbit in $\underline{O}(\mathbb{Z})$ with coordinates only differing by signs. For orbits in $\underline{O}(\mathbb{Z})^\circ$, the four u -coordinates do not vanish; there are exactly two integral points in each such $\underline{T}(\mathbb{Z})$ -orbit with $u > 0$ and all $u_j > 0$, and these two points have the same v_j -coordinates and opposite non-zero w_j -coordinates. Summarizing the above discussion, we have shown that there is a bijection between the set of such pairs in \mathcal{A}_0 and $\underline{X}(\mathbb{Z})^\circ = \underline{W}(\mathbb{Z})^\circ$ where the latter set may be identified with the pairs $\pm(x_1, x_2, x_3, y_1, y_2, y_3)$ of primitive sextuples of integers satisfying (1.1) and (1.3). The map from $\underline{O}(\mathbb{Z})^\circ$ to $\underline{W}(\mathbb{Z})^\circ$ comes from $\underline{f} \circ \varphi$ and is thus given by (4.3) and (4.6). As the signs of $(x_1, x_2, x_3, y_1, y_2, y_3)$ are opposite for the two tuples in \mathcal{A}_0 in the same $\underline{T}(\mathbb{Z})$ -orbit, we have established the desired bijection.

5. PEYRE'S CONJECTURE

The aim of this chapter is to formulate Peyre's conjecture on the asymptotic behaviour of $N(P)$. As the fourfold $W \subset \mathbb{P}^5$ defined by (1.1) is singular, we cannot refer to the original conjectures of Manin [FMT] and Peyre [Pe1] for Fano varieties. But as $f : X \rightarrow W$ restricts to an isomorphism from X° to W° , we obtain that

$$N(P) = |\{x \in X^\circ(\mathbb{Q}) : (H \circ f)(x) \leq P\}|$$

where the height function $H \circ f : X(\mathbb{Q}) \rightarrow \mathbb{N}$ is anticanonical, as f is crepant. Since X is an “almost Fano” variety (see Lemma 6), we may refer to the “formule empirique” in [Pe2, 5.1] for anticanonical counting functions on such varieties. This formula predicts that

$$(5.1) \quad N(P) \sim \Theta_H(X) P(\log P)^{\text{rk Pic}(X)-1}$$

where $\Theta_H(X) = \alpha(X)\tau_H(X)$ for a suitable adelic Tamagawa volume $\tau_H(X)$ of $X(\mathbf{A})$ and $\alpha(X)$ as in (3.8). Peyre demands for his formula to hold that the cohomological Brauer group $Br'(\bar{X}) = H_{\text{et}}^2(\bar{X}, \mathbb{G}_m)$ vanishes which is true for our rational fourfold since $Br'(\bar{X})$ is a birational invariant [Gr2, Thm 7.1] and $Br'(\mathbb{P}_{\mathbb{Q}}^n) = 0$. There is also one hypothesis on $C_{\text{eff}}(X)$ in [Pe2, 3.3] which is satisfied thanks to Lemma 4(ii). Finally, Peyre assumes that there are no weakly accumulating subsets (see [Pe2, (3.1)]) on X° for (5.1) to be valid. This will follow from our asymptotic formula, but is expected here because of the group structure on $G = X^\circ$.

The main goal of this chapter is to define and compute $\tau_H(X)$ for our fourfold X , which together with the previous results on $\text{Pic}(X)$ in (3.7) and $\alpha(X)$ in Lemma 5 gives an explicit conjecture for the asymptotic formula of $N(P)$.

For $i = 4, 5, 6$ it will be convenient to set $x_i = y_{i-3}$ and $F(x_1, \dots, x_6) = x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2$. For $1 \leq i \leq 6$ we write $\mathbb{P}_{(i)}^5 \subset \mathbb{P}^5$, $\Xi_{(i)} \subset \Xi$ (with Ξ as in the previous chapter), $W_{(i)} \subset W$ and $X_{(i)} \subset X$ for the principal open subsets where $x_i \neq 0$, and we introduce the affine coordinates $x_j^{(i)} = x_j/x_i$, $j \neq i$ for $\mathbb{P}_{(i)}^5 = \mathbb{A}^5$ and $\Xi_{(i)} \subset \mathbb{P}_{(i)}^5 \times \mathbb{P}^2 \times \mathbb{P}^2$. Then $W_{(i)}$ is the affine hypersurface in $\mathbb{P}_{(i)}^5$ defined by $F_i(x_1^{(i)}, \dots, x_i^{(i)}, \dots, x_6^{(i)}) = F(x_1^{(i)}, \dots, 1, \dots, x_6^{(i)})$.

To define $\tau_H(X)$ we need another description of the height function $H \circ f : X(\mathbb{Q}) \rightarrow \mathbb{N}$ in terms of an adelic metric on the anticanonical sheaf ω_X^{-1} . This adelic metric will be defined by means of global sections of $\omega_X^{-1} = f^*(\omega_W^{-1})$, which are inverse images of global sections on ω_W^{-1} . If $s \in \Gamma(U, L)$ is a local section of an O_W -module L , we shall write $f^*(s)$ for the local section $f^{-1}(s) \otimes_{f^{-1}O_W} 1 \in \Gamma(f^{-1}(U), f^*(L))$ of $f^*(L) = f^{-1}(L) \otimes_{f^{-1}O_W} O_X$.

The global sections of ω_W^{-1} and ω_X^{-1} that we shall use are dual to certain 4-forms on W and X . These 4-forms are given by Poincaré residues of rational 5-forms on \mathbb{P}^5 and Ξ . To control the rational 5-forms on \mathbb{P}^5 and Ξ , we need the following lemma from the theory of toric varieties [Fu2, p. 86].

Lemma 12. *Let V be a non-singular n -dimensional toric variety with torus U and $\omega_V(\sum_{k=1}^r D_k)$ be the sheaf of n -forms on V with at most simple poles along all irreducible components D_1, \dots, D_r of $\delta V = V \setminus U$. Then there is a global section $s_V \in \Gamma(V, \omega_V(\sum_{k=1}^r D_k))$ such that $s_V = \pm \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n}$ on U for any set of n characters $\chi_i : U \rightarrow \mathbb{G}_m$ that form a basis of $M = \text{Hom}(U, \mathbb{G}_m)$. The section s_V generates the O_V -module $\omega_V(\sum_{k=1}^r D_k)$.*

We now apply this lemma to the torus U given by the cokernel of the diagonal inclusion of \mathbb{G}_m in \mathbb{G}_m^6 , and to the toric fivefolds $V = \mathbb{P}^5$ and $V = \Xi$. If we let $p_1 : \Xi \rightarrow \mathbb{P}^5$ be the restriction of $\mathbb{P}^5 \times \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^5$ to Ξ , then p_1 gives an isomorphism between the open subsets $U_\Xi \subset \Xi$ and $U_{\mathbb{P}^5} \subset \mathbb{P}^5$, where $x_i \neq 0$ for all $1 \leq i \leq 6$. If we make the obvious identifications of these two open subsets with U , then the group law on U extends to actions of U on Ξ and \mathbb{P}^5 such that p_1 is U -equivariant.

In particular (see [Re3, p. 41]), if $V = \mathbb{P}^5$ and H_k , $1 \leq k \leq 6$, are the coordinate planes of \mathbb{P}^5 , then there is a global nowhere vanishing section $s_{\mathbb{P}^5}$ of $\omega_{\mathbb{P}^5}(\sum_{k=1}^6 H_k)$, such that the restriction $s_{\mathbb{P}^5}^{(i)}$ of $s_{\mathbb{P}^5}$ to $\mathbb{P}_{(i)}^5$ is equal to

$$s_{\mathbb{P}^5}^{(i)} = (-1)^i \frac{dx_1^{(i)}}{x_1^{(i)}} \wedge \dots \wedge \widehat{\frac{dx_i^{(i)}}{x_i^{(i)}}} \wedge \dots \wedge \frac{dx_6^{(i)}}{x_6^{(i)}} \in \Gamma(\mathbb{P}_{(i)}^5, \omega_{\mathbb{P}^5}(\sum_{k=1}^6 H_k)).$$

On the other hand, if $V = \Xi$, then we conclude from the proof of Lemma 10 that there are ten irreducible components of $\Xi \setminus U$ corresponding to the ten edges of the fan of Ξ and to the ten coordinate hyperplane sections of the universal torsor $\Omega \subset \mathbb{A}^{10}$. We let $D(\xi_i)$, $0 \leq i \leq 3$, be the image in Ξ of the subset of Ω defined by $\xi_i = 0$, and we let $D(u_j)$ and $D(w_j)$, $1 \leq j \leq 3$, be the prime divisors on Ξ defined in the same way. Then by Lemma 12 there is a global section s_Ξ on $\omega_\Xi(E)$ for $E = \sum_{j=0}^3 D(\xi_j) + \sum_{j=1}^3 (D(u_j) + D(w_j))$ such that the restriction of s_Ξ to the open subset $\Xi_{(i)}$ of Ξ where $x_i \neq 0$ is given by

$$s_\Xi^{(i)} = (-1)^i \frac{dx_1^{(i)}}{x_1^{(i)}} \wedge \dots \wedge \widehat{\frac{dx_i^{(i)}}{x_i^{(i)}}} \wedge \dots \wedge \frac{dx_6^{(i)}}{x_6^{(i)}} \in \Gamma(\Xi_{(i)}, \omega_\Xi(E)).$$

Now let

$$\begin{aligned} \omega_i &= \frac{x_1 x_2 x_3 x_4 x_5 x_6}{x_i^3 F} s_{\mathbb{P}^5} \in \Gamma(\mathbb{P}^5, \omega_{\mathbb{P}^5}(W + 3H_i)), \\ \varpi_i &= \frac{x_1 x_2 x_3 x_4 x_5 x_6}{x_i^3 F} s_\Xi \in \Gamma(\Xi, \omega_\Xi(X + 3p_1^* H_i)). \end{aligned}$$

Then on the open subsets where $x_i \neq 0$, we have for $1 \leq i \leq 6$ that

$$\begin{aligned} \omega_i &= \frac{(-1)^i}{F_i} dx_1^{(i)} \wedge \dots \wedge \widehat{dx_i^{(i)}} \wedge \dots \wedge dx_5^{(i)} \in \Gamma(\mathbb{P}_{(i)}^5, \omega_{\mathbb{P}^5}(W)), \\ \varpi_i &= \frac{(-1)^i}{F_i} dx_1^{(i)} \wedge \dots \wedge \widehat{dx_i^{(i)}} \wedge \dots \wedge dx_5^{(i)} \in \Gamma(\Xi_{(i)}, \omega_{\mathbb{P}^5}(X)). \end{aligned} \tag{5.2}$$

We now consider Poincaré residues of these forms. The Poincaré residue map is usually given as a homomorphism $\Omega_V^n(W) \rightarrow i_* \Omega_W^{n-1}$ for the inclusion map i of a non-singular hypersurface $W \subset V$

in an n -dimensional non-singular variety (cf. [Re3, p. 89], for example). More generally, one can also use Poincaré residues to define local sections on the canonical sheaf ω_W of an arbitrary normal hypersurface (cf. [We]) as one still gets regular $(n-1)$ -forms on the non-singular locus W_{ns} of W and since $\omega_W = j_*\Omega_{W_{ns}}^{n-1}$ for the open embedding $j : W_{ns} \rightarrow W$. For our singular hypersurface $i : W \rightarrow \mathbb{P}^5$ we therefore have a unique homomorphism $\text{Res} : \omega_{\mathbb{P}^5}(W) \rightarrow i_*\omega_W$ of $\mathcal{O}_{\mathbb{P}^5}$ -modules, which sends $\omega_i \in \Gamma(\mathbb{P}_{(i)}^5, \omega_{\mathbb{P}^5}(W))$ to the section $\text{Res}(\omega_i) \in \Gamma(\mathbb{P}_{(i)}^5, i_*\omega_W) = \Gamma(W_{(i)}, \omega_W)$ which at the open subset of $W_{(i)}$ where $\partial F_i / \partial x_j^{(i)} \neq 0$ is given by

$$(5.3) \quad \text{Res}(\omega_i) = \begin{cases} \frac{(-1)^{i+k}}{\partial F_i / \partial x_k^{(i)}} dx_1^{(i)} \wedge \dots \wedge \widehat{dx_k^{(i)}} \wedge \dots \wedge \widehat{dx_k^{(i)}} \wedge \dots \wedge dx_6^{(i)} & \text{if } i < k, \\ \frac{(-1)^{i+k-1}}{\partial F_i / \partial x_k^{(i)}} dx_1^{(i)} \wedge \dots \wedge \widehat{dx_k^{(i)}} \wedge \dots \wedge \widehat{dx_i^{(i)}} \wedge \dots \wedge dx_6^{(i)} & \text{if } k < i. \end{cases}$$

Similarly, for the inclusion $\iota : X \subset \Xi$, we note that $X_{(i)} \subset \Xi_{(i)}$ is defined by F_i on the open subset of $\Xi_{(i)}$ where two of y_1, y_2, y_3 are different from zero. As $\varpi_i \in \Gamma(\Xi_{(i)}^5, \omega_{\Xi}(X))$ is given by the same 5-form as in (5.2), we obtain similarly that

$$(5.4) \quad \text{Res}(\varpi_i) = \begin{cases} \frac{(-1)^{i+k}}{\partial F_i / \partial x_k^{(i)}} dx_1^{(i)} \wedge \dots \wedge \widehat{dx_k^{(i)}} \wedge \dots \wedge \widehat{dx_k^{(i)}} \wedge \dots \wedge dx_6^{(i)} & \text{if } i < k, \\ \frac{(-1)^{i+k-1}}{\partial F_i / \partial x_k^{(i)}} dx_1^{(i)} \wedge \dots \wedge \widehat{dx_k^{(i)}} \wedge \dots \wedge \widehat{dx_i^{(i)}} \wedge \dots \wedge dx_6^{(i)} & \text{if } k < i \end{cases}$$

on the open subset of $X_{(i)}$ where $(y_1 y_2, y_1 y_3, y_2 y_3) \neq (0, 0, 0)$.

Lemma 13. (i) *The section $\text{Res}(\omega_i)$ extends uniquely to a global nowhere vanishing section of $\omega_W(3(H_i \cap W))$.*
(ii) *The section $\text{Res}(\varpi_i)$ extends uniquely to a global nowhere vanishing section of $\omega_X(3f^*(H_i \cap W))$.*
(iii) *The section $f^*(\text{Res}(\omega_i)) \in \Gamma(X, f^*\omega_W(3(H_i \cap W)))$ is sent to $\text{Res}(\varpi_i)$ under the natural homomorphism from $f^*\omega_W(3(W \cap H_i))$ to $\omega_X(3f^*(H_i \cap W))$.*

Proof. To prove (i) and (ii), we make use of the adjunction formula (see [GH, pp. 146-147]). In this way we obtain isomorphisms $i^*\omega_{\mathbb{P}^5}(W) \rightarrow \omega_W$ and $\iota^*\omega_{\Xi}(X) \rightarrow \omega_X$ adjoint to the Poincaré residue maps. These maps induce in turn isomorphisms $i^*\omega_{\mathbb{P}^5}(W + 3H_i) \rightarrow \omega_W(3(H_i \cap W))$ and $\iota^*\omega_{\Xi}(X + 3p_1^*H_i) \rightarrow \omega_X(3f^*(H_i \cap W))$, which send $i^*\omega_i$ to $\text{Res}(\omega_i)$ and $\iota^*\varpi_i$ to $\text{Res}(\varpi_i)$. Hence it suffices for the proof of (i) and (ii) to show that $\omega_i = \frac{x_1 x_2 x_3 x_4 x_5 x_6}{x_i^3 F} s_{\mathbb{P}^5}$ generates the $\mathcal{O}_{\mathbb{P}^5}$ -module $\omega_{\mathbb{P}^5}(W + 3H_i)$ and that $\varpi_i = \frac{x_1 x_2 x_3 x_4 x_5 x_6}{x_i^3 F} s_{\Xi}$ generates the \mathcal{O}_{Ξ} -module $\omega_{\Xi}(X + 3p_1^*H_i)$. This follows from the last assertion of Lemma 12. Part (iii) follows from (5.3) and (5.4).

Thanks to Lemma 13, we may now define global nowhere vanishing sections $\tau_i = \text{Res}(\omega_i)^{-1}$ of $\omega_W^{-1}(-3(H_i \cap W))$ and $\sigma_i = \text{Res}(\varpi_i)^{-1}$ of $\omega_X^{-1}(-3f^*(H_i \cap W))$. We will regard them as anticanonical global sections and use the following result to define v -adic norms and measures.

Lemma 14. (i) *The section $\tau_i \in \Gamma(W, \omega_W^{-1})$ does not vanish anywhere on $W_{(i)}$.*
(ii) *The section $\sigma_i \in \Gamma(X, \omega_X^{-1})$ does not vanish anywhere on $X_{(i)}$.*
(iii) *The section $f^*\tau_i$ is mapped to σ_i under the canonical isomorphism from $f^*\omega_W^{-1}$ to ω_X^{-1} .*

Proof. This is an immediate consequence of the previous lemma since $\omega_W^{-1}(-3(H_i \cap W)) = \omega_W^{-1}$ on $W_{(i)}$ and $\omega_X^{-1}(-3f^*(H_i \cap W)) = \omega_X^{-1}$ on $X_{(i)}$.

In the following we shall use the standard absolute values $|\cdot|_v : \mathbb{Q}_v \rightarrow [0, \infty)$ for the places v of \mathbb{Q} (including the archimedean place). As τ_i vanishes nowhere on $W_{(i)}$, we obtain for each place v a v -adic norm on ω_W^{-1} by letting

$$\|\tau(w_v)\|_v = \min_j \left| \frac{\tau}{\tau_j}(w_v) \right|_v = \min_j |(\tau \text{Res}(\omega_j))(w_v)|_v$$

for a local section τ of ω_W^{-1} defined at $w_v \in W(\mathbb{Q}_v)$ and where the minimum is taken over all $j \in \{1, \dots, 6\}$ such that $\tau_j(w_v) \neq 0$. This definition is the same as in [Pe1, pp. 107-108], although it is called a v -adic metric there. For more on v -adic norms on invertible sheaves, see also [Sa, Chapter 1].

As σ_i vanishes nowhere on $X_{(i)}$, we obtain in the same way a v -adic norm on ω_X^{-1} for each place v of \mathbb{Q} by letting

$$(5.5) \quad \|\sigma(x_v)\|_v = \min_j \left| \frac{\sigma}{\sigma_j}(x_v) \right|_v = \min_j |(\sigma \text{Res}(\varpi_j))(x_v)|_v$$

for a local section σ of ω_X^{-1} defined at $x_v \in X(\mathbb{Q}_v)$ and where again the minimum is taken over all $j \in \{1, \dots, 6\}$ with $\sigma_j(x_v) \neq 0$.

Lemma 15. (i) *Let $w \in W(\mathbb{Q})$ and let τ be a local section of ω_W^{-1} with $\tau(w) \neq 0$. Then*

$$(5.6) \quad H(w) = \prod_{\text{all } v} \|\tau(w)\|_v^{-1}.$$

(ii) *Let $x \in X(\mathbb{Q})$ and let σ be a local section of ω_X^{-1} with $\sigma(x) \neq 0$. Then*

$$(5.7) \quad H(f(x)) = \prod_{\text{all } v} \|\sigma(x)\|_v^{-1}.$$

Proof. (i) As $\prod_v |\alpha|_v = 1$ for $\alpha \in \mathbb{Q}^*$ it suffices to show (5.6) for one such local section τ . So let $C \in \mathbb{Q}[x_1, x_2, x_3, x_4, x_5, x_6]$ be a cubic form with $C(w) \neq 0$ and $\tau = \frac{C}{x_j^3} \tau_j$ for j with $x_j(w) \neq 0$. Then

$$\|\tau(w)\|_v^{-1} = \max_{1 \leq j \leq 6} \left| \frac{\tau_j}{\tau}(w) \right|_v = \max_{1 \leq j \leq 6} \left| \frac{x_j^3}{C}(w) \right|_v$$

which immediately gives the desired formula for $H(w)$.

To prove (ii), we use the canonical isomorphism $f^*(\omega_W^{-1}) = \omega_X^{-1}$ and choose σ to be the image of $f^*(\tau)$ for some local section τ of ω_W^{-1} where $\tau(w) \neq 0$ for $w = f(x)$. It follows from Lemma 14(iii) that $\|\sigma(x)\|_v = \|\tau(w)\|_v$ for each v , so that (5.7) follows from (5.6).

We now apply Peyre's definition [Pe1, (2.2.1)] of a measure μ_v on $X(\mathbb{Q}_v)$ associated to a v -adic norm on ω_X^{-1} . Let $|\text{Res}(\varpi_i)|_v$ be the v -adic density on $X_{(i)}(\mathbb{Q}_v)$ of the volume form $\text{Res}(\varpi_i)$ on $X_{(i)}$. Then for our particular v -adic norm $\|\cdot\|_v$ defined in (5.5), we get the measure where

$$(5.8) \quad \mu_v(N_v) = \int_{N_v} \frac{|\text{Res}(\varpi_i)|_v}{\max_{1 \leq j \leq 6} |\sigma_j \text{Res}(\varpi_i)|_v} = \int_{N_v} \frac{|\text{Res}(\varpi_i)|_v}{\max_{1 \leq j \leq 6} |(x_j/x_i)^3|_v}$$

for a Borel subset N_v of $X_{(i)}(\mathbb{Q}_v)$.

To get a more explicit description of μ_v , let us write $t_j = x_j^{(6)} = x_j/x_6$. Then, by (5.2) and (5.4), we have that

$$\omega_6 = \frac{1}{F_6} dt_1 \wedge dt_2 \wedge dt_3 \wedge dt_4 \wedge dt_5, \quad \text{Res}(\omega_6) = \frac{(-1)^{k-1}}{\partial F_6 / \partial t_k} dt_1 \wedge \dots \wedge \widehat{dt_k} \wedge \dots \wedge dt_5$$

for any $k = 1, 2, 3, 4, 5$ and $F_6(t_1, t_2, t_3, t_4, t_5) = t_1 t_5 + t_2 t_4 + t_3 t_4 t_5$. For instance, choosing $k = 3$, we obtain

$$(5.9) \quad \mu_v(N_v) = \int_{N_v} \frac{dt_1 dt_2 dt_4 dt_5}{|t_4 t_5|_v \max(|t_1|_v^3, |t_2|_v^3, |\frac{t_1}{t_4} + \frac{t_2}{t_5}|_v^3, |t_4|_v^3, |t_5|_v^3, 1)}$$

for any Borel subset N_v of $\bigcap_{3 \leq i \leq 6} X_{(i)}(\mathbb{Q}_v)$. Here and elsewhere we assume that the underlying Haar measure on \mathbb{Q}_v is the usual Lebesgue measure if $\mathbb{Q}_v = \mathbb{R}$, and that it is normalized by $\int_{\mathbb{Z}_p} dx = 1$ if v is p -adic.

Now let $L_p(s, \text{Pic}(\overline{X})) = \det(1 - p^{-s} \text{Fr}_p \mid \text{Pic}(X_{\overline{\mathbb{F}}_p}) \otimes \mathbb{Q})^{-1}$ for a prime p . As $\text{Pic}(X_{\overline{\mathbb{F}}_p}) = \mathbb{Z}^5$ with trivial Galois action, we get that

$$L(s, \text{Pic}(\overline{X})) = \prod_{\text{all } p} L_p(s, \text{Pic}(\overline{X})) = \prod_{\text{all } p} (1 - p^{-s})^{-1} = \zeta(s)^5$$

for $s \in \mathbb{C}$ with $\text{Re } s > 1$. In particular, $\lim_{s \rightarrow 1} (s - 1)^5 L(s, \text{Pic}(\overline{X})) = 1$ and $L_p(1, \text{Pic}(\overline{X}))^{-1} = (\frac{p-1}{p})^5$. For our particular fourfold X , Peyre's Tamagawa measure μ_H on $X(\mathbf{A}) = X(\mathbb{R}) \times \prod_p X(\mathbb{Q}_p)$ (see [Pe2, Def. 4.6]) is therefore given by $\mu_H = \mu_\infty \times \prod_p (\frac{p-1}{p})^5 \mu_p$, and it is shown in [Pe2] that this gives a well-defined measure on $X(\mathbf{A})$. As $X(\mathbb{Q})$ is dense in $X(\mathbf{A})$, we thus have

$$(5.10) \quad \tau_H(X) = \mu_H(X(\mathbf{A})) = \mu_\infty(X(\mathbb{R})) \prod_{\text{all } p} \left(\frac{p-1}{p} \right)^5 \mu_p(X(\mathbb{Q}_p))$$

by [Pe2, Def. 4.8], and it remains to determine the local volumes $\mu_\infty(X(\mathbb{R}))$ and $\mu_p(X(\mathbb{Q}_p))$.

To compute $\mu_\infty(X(\mathbb{R}))$, let $N_\infty(\mathbb{R}) = \bigcap_{3 \leq i \leq 6} X_{(i)}(\mathbb{R})$. Then, $\mu_\infty(X(\mathbb{R})) = \mu_\infty(N_\infty(\mathbb{R}))$ by Sard's theorem. Hence, by (5.9), we obtain

$$\mu_\infty(X(\mathbb{R})) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dt_1 dt_2 dt_4 dt_5}{|t_4 t_5| \max(|t_1|^3, |t_2|^3, |\frac{t_1}{t_4} + \frac{t_2}{t_5}|^3, |t_4|^3, |t_5|^3, 1)}.$$

It is a long, but elementary and straightforward calculation to check that

$$(5.11) \quad \mu_\infty(X(\mathbb{R})) = 12(\pi^2 + 24 \log 2 - 3).$$

We proceed to compute $\mu_p(X(\mathbb{Q}_p))$. Let $\Omega \subset \mathbb{A}^{10}$ and $\varphi : \Omega \rightarrow \Xi$ be as in Lemma 10. Then $O = \varphi^{-1}(X)$, and the structure morphism of the X -torsor O is given by the restriction $\varphi_O : O \rightarrow X$ of φ to O (see Theorem 7). Hence $O \subset \Omega$ is the hypersurface defined by $\Phi = \xi_1 u_1 w_2 w_3 + \xi_2 u_2 w_1 w_3 + \xi_3 u_3 w_1 w_2$, cf. (4.16). By Lemma 12, there is a rational 10-form s_Ω on the toric variety $\Omega \subset \mathbb{A}^{10}$ defined by

$$s_\Omega = \frac{d\xi_0}{\xi_0} \wedge \frac{d\xi_1}{\xi_1} \wedge \frac{d\xi_2}{\xi_2} \wedge \frac{d\xi_3}{\xi_3} \wedge \frac{du_1}{u_1} \wedge \frac{du_2}{u_2} \wedge \frac{du_3}{u_3} \wedge \frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2} \wedge \frac{dw_3}{w_3}.$$

For $1 \leq i \leq 6$ we let $\varpi_i^\Omega = \frac{x_1 x_2 x_3 x_4 x_5 x_6}{x_i^3 F} s_\Omega$. Finally, we define

$$(5.12) \quad \varpi^\Omega = \frac{1}{\Phi} d\xi_0 \wedge d\xi_1 \wedge d\xi_2 \wedge d\xi_3 \wedge du_1 \wedge du_2 \wedge du_3 \wedge dw_1 \wedge dw_2 \wedge dw_3.$$

Then, from $F = \xi_0^2 u_1 u_2 u_3 \Phi$, we conclude that $\varpi_i^\Omega = \varpi^\Omega / x_i^3$ on $\Omega_{(i)} = \varphi^{-1}(\Xi_{(i)})$ for $1 \leq i \leq 6$, where x_i and $x_{i+3} = y_i$, $1 \leq i \leq 3$, now denote the *affine* coordinates given by the expressions in (4.15).

The following construction works for an arbitrary place v , although in the present situation we are only interested in non-archimedean places. Any $\varpi_i^\Omega \in \Gamma(\Omega_{(i)}, \omega_\Omega(O))$ has a Poincaré residue $\text{Res}(\varpi_i^\Omega) \in \Gamma(O_{(i)}, \omega_O)$ on $O_{(i)} = \varphi^{-1}(X_{(i)})$. We may now, just as in (5.8), use the six local volume forms $\text{Res}(\varpi_i^\Omega)$ to construct a v -adic measure m_v on $O(\mathbb{Q}_v)$ by letting

$$m_v(M_v) = \int_{M_v} \frac{|\text{Res}(\varpi_i^\Omega)|_v}{\max_{1 \leq j \leq 6} |(x_j/x_i)^3|_v}$$

for a Borel subset M_v of $O_{(i)}(\mathbb{Q}_v)$. The connection between m_p and μ_p for a prime p will become clear in Lemma 18 below. To start with, we consider the relative canonical sheaves $\omega_{\Omega/\Xi}$, $\omega_{O/X}$ and apply the following result.

Lemma 16. *Let $E^\Omega = \varphi^*E \in \text{Div}(\Omega)$ be the sum of the ten prime divisors of Ω defined by the ten coordinate hyperplanes of \mathbb{A}^{10} . Then there is a unique global nowhere vanishing section $s_{\Omega/\Xi} \in \Gamma(\Omega, \omega_{\Omega/\Xi})$ such that $s_\Omega = s_{\Omega/\Xi} \otimes \varphi^*s_\Xi$ under the natural isomorphism $\omega_\Omega(E^\Omega) = \omega_{\Omega/\Xi} \otimes \varphi^*\omega_\Xi(E)$. Moreover, if we let $\iota_O : O \rightarrow \Omega$ be the inclusion map, $s_{O/X} \in \Gamma(O, \omega_{O/X})$ be the image of $\iota_O^*s_{\Omega/\Xi} \in \Gamma(O, \iota_O^*\omega_{\Omega/\Xi})$ under the functorial isomorphism from $\iota_O^*\omega_{\Omega/\Xi}$ to $\omega_{O/X}$ and $s_{O/X}^{(i)}$ be the restriction of $s_{O/X}$ to $O_{(i)}$, then*

$$(5.13) \quad \text{Res}(\omega_i^\Omega) = s_{O/X}^{(i)} \otimes \varphi_O^* \text{Res}(\varpi_i)$$

for $1 \leq i \leq 6$ under the canonical isomorphism $\omega_O = \omega_{O/X} \otimes \varphi_O^*\omega_X$.

Proof. The isomorphism between $\omega_\Omega(E^\Omega)$ and $\omega_{\Omega/\Xi} \otimes \varphi^*\omega_\Xi(E)$ is induced by the canonical isomorphism between ω_Ω and $\omega_{\Omega/\Xi} \otimes \varphi^*\omega_\Xi$, and the first statement is obvious as s_Ω (resp. φ^*s_Ξ) is a global generator of $\omega_\Omega(E^\Omega)$ (resp. $\varphi^*\omega_\Xi(E)$). The second statement follows from a functoriality property of Poincaré residues, which says that there is a natural commutative diagram of isomorphisms of O_O -modules

$$\begin{array}{ccc} \iota_O^*\omega_\Omega(E^\Omega) & \longrightarrow & \iota_O^*(\omega_{\Omega/\Xi}) \otimes \iota_O^*\varphi^*\omega_\Xi(E) \\ \downarrow & & \downarrow \\ \omega_O & \longrightarrow & \omega_{O/X} \otimes \varphi_O^*\omega_X \end{array}$$

where the first vertical map is given by the adjoint Poincaré residue map for $\iota_O : O \rightarrow \Omega$ and the second vertical map makes use of the isomorphism from $\iota_O^*\varphi^*\omega_\Xi(E) = \varphi_O^*\iota^*\omega_\Xi(E)$ to $\varphi_O^*\omega_X$ induced by the adjoint Poincaré residue map for $\iota : X \rightarrow \Xi$.

We now apply this result to the v -adic analytic manifolds associated to O and X and refer to [Se, Ch. III] for basic definitions and properties of such manifolds and to [Sa, Ch. 3] for the notion of torsors over v -adic analytic manifolds.

Lemma 17. (i) *The map $\varphi_{O,v} : O(\mathbb{Q}_v) \rightarrow X(\mathbb{Q}_v)$ induced by φ_O is a submersion of v -adic analytic manifolds, which makes $O(\mathbb{Q}_v)$ an analytic $X(\mathbb{Q}_v)$ -torsor under $T(\mathbb{Q}_v)$.*
(ii) *The relative volume form $s_{O/X} \in \Gamma(O, \omega_{O/X})$ defines v -adic measures on the fibres of $\varphi_{O,v}$ which yields a linear functional $\Lambda_v : C_c(O(\mathbb{Q}_v)) \rightarrow C_c(X(\mathbb{Q}_v))$ when we integrate along the fibres of $\varphi_{O,v}$.*
(iii) *If $\beta_v \in C_c(O(\mathbb{Q}_v))$, then $\int_{O(\mathbb{Q}_v)} \beta_v m_v = \int_{X(\mathbb{Q}_v)} \Lambda_v(\beta_v) \mu_v$.*

Proof. For (i), see [Sa, pp. 126-127]. To obtain (ii) and (iii), use [Sa, Theorem 1.22] and (5.13).

We may now reinterpret the p -adic factor of Peyre's constant $\Theta_H(X)$ as a p -adic density of the universal torsor over X .

Lemma 18. *Let \underline{Q} be the scheme defined in Section 4.2. Then $m_p(\underline{Q}(\mathbb{Z}_p)) = (\frac{p-1}{p})^5 \mu_p(X(\mathbb{Q}_p))$ for any prime p .*

Proof. We embed $\underline{Q}(\mathbb{Z}_p)$ as an open subset of $\underline{Q}(\mathbb{Q}_p) = O(\mathbb{Q}_p)$ and let $\chi_p : O(\mathbb{Q}_p) \rightarrow \{0, 1\}$ be the characteristic function of $\underline{Q}(\mathbb{Z}_p)$. Then $\chi_p \in C_c(O(\mathbb{Q}_p))$ and $m_p(\underline{Q}(\mathbb{Z}_p)) = \int_{X(\mathbb{Q}_p)} \Lambda_p(\chi_p) \mu_p$ by the previous lemma. It is therefore enough to show that $\Lambda_p(\chi_p) \in C_c(X(\mathbb{Q}_p))$ has value $(\frac{p-1}{p})^5$ at all points of $\underline{X}(\mathbb{Z}_p) = X(\mathbb{Q}_p)$. But it is clear that the decomposition $s_\Omega = s_{\Omega/\Xi} \otimes \varphi^*s_\Xi$ may be carried out over \mathbb{Z} such that $s_{\Omega/\Xi}$ extends to a \underline{T} -equivariant generator of $\omega_{\underline{\Omega}/\underline{\Xi}}$ and $s_{O/X}$ to a \underline{T} -equivariant generator $s_{\underline{O}/\underline{X}}$ of $\omega_{\underline{O}/\underline{X}}$. If P is a \mathbb{Z}_p -point on \underline{X} and $\underline{Q}_P \rightarrow P$ the base extension of $\underline{Q} \rightarrow \underline{X}$, then $s_{\underline{O}/\underline{X}}$ will therefore pull back to a $\underline{T}_{\mathbb{Z}_p}$ -equivariant global section $s_{\underline{Q}_P}$ on $\omega_{\underline{Q}_P/\mathbb{Z}_p}$. As the torsor over P is trivial and $\underline{T} \cong \mathbb{G}_{m, \mathbb{Z}}^5$, there are affine coordinates (t_1, \dots, t_5) for the affine \mathbb{Z}_p -scheme \underline{Q}_P such

that $s_{\underline{Q}_P} = \frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_5}{t_5}$. Hence

$$\Lambda_p(\chi_p)(P) = \int_{\underline{Q}_P(\mathbb{Z}_p)} |s_{\underline{Q}_P}| = \prod_{1 \leq i \leq 5} \int_{\mathbb{Z}_p^*} \frac{dt_i}{t_i} = \left(\frac{p-1}{p} \right)^5,$$

and we are done.

To compute $m_p(\underline{Q}(\mathbb{Z}_p))$, we give an alternative definition of m_p . As $\omega^\Omega \in \Gamma(\Omega, \omega_\Omega(O))$ by (5.12), it has a residue form $\text{Res}(\omega^\Omega) \in \Gamma(O, \omega_O)$. If we again let x_i , $1 \leq i \leq 6$, denote the affine coordinates given by the expressions in (4.15), then ω^Ω restricts to $x_i^3 \text{Res}(\omega_i^\Omega)$ on $O_{(i)}$ such that

$$(5.14) \quad m_p(M_p) = \int_{M_p} \frac{|\text{Res}(\omega^\Omega)|_p}{\max_{1 \leq j \leq 6} |x_j^3|_p} = \int_{M_p} \frac{d\xi \, d\mathbf{u} \, d\mathbf{w} / d\Phi}{\max_{1 \leq j \leq 6} |x_j^3|_p}$$

for Borel subsets M_p of $O(\mathbb{Q}_p)$ and for the p -adic density $|\text{Res}(\omega^\Omega)|_p$ of $\text{Res}(\omega^\Omega)$, which we also denote by $d\xi \, d\mathbf{u} \, d\mathbf{w} / d\Phi$. Here we have written $d\xi = d\xi_0 d\xi_1 d\xi_2 d\xi_3$, $d\mathbf{u} = du_1 du_2 du_3$ and $d\mathbf{w} = dw_1 dw_2 dw_3$ for notational simplicity.

Lemma 19. *One has*

$$m_p(\underline{Q}(\mathbb{Z}_p)) = \frac{|\underline{Q}(\mathbb{F}_p)|}{p^{\dim O}} = \frac{(p-1)^5(p^2+p+1)(p^2+4p+1)}{p^9}.$$

Proof. For $P \in \underline{Q}(\mathbb{Z}_p)$ we may find some $j \in \{1, \dots, 6\}$ such that p does not divide $x_j(P)$ (cf. (4.15)), since φ restricts to a morphism from $\underline{Q}_{\mathbb{F}_p}$ to $\Xi_{\mathbb{F}_p} \subset \mathbb{P}_{\mathbb{F}_p}^5 \times \mathbb{P}_{\mathbb{F}_p}^2 \times \mathbb{P}_{\mathbb{F}_p}^2$ modulo p . We conclude $\max_{1 \leq j \leq 6} |x_j^3|_p = 1$ for $P \in \underline{Q}(\mathbb{Z}_p) \subset \underline{Q}(\mathbb{Z}_p)$ and

$$m_p(\underline{Q}(\mathbb{Z}_p)) = \int_{\underline{Q}(\mathbb{Z}_p)} \frac{d\xi \, d\mathbf{u} \, d\mathbf{w}}{d\Phi}$$

by (5.14). It is easy to see that this measure coincides with the p -adic model measure defined in [Sa, 2.9]. As \underline{Q} is smooth over \mathbb{Z} , we may thus apply [Sa, Cor. 2.15] and conclude that $m_p(\underline{Q}(\mathbb{Z}_p)) = |\underline{Q}(\mathbb{F}_p)|/p^{\dim O}$. To determine $|\underline{Q}(\mathbb{F}_p)|$, we note that the $\underline{X}_{\mathbb{F}_p}$ -torsor $\underline{Q}_{\mathbb{F}_p}$ under $\underline{T}_{\mathbb{F}_p}$ is locally trivial, such that $|\underline{Q}(\mathbb{F}_p)|/p^{\dim O} = |\underline{T}(\mathbb{F}_p)| \cdot |\underline{X}(\mathbb{F}_p)|/p^9$. To finish we note that $|\underline{T}(\mathbb{F}_p)| = (p-1)^5$ and $|\underline{X}(\mathbb{F}_p)| = (p^2+p+1)(p^2+4p+1)$ as $\underline{X}_{\mathbb{F}_p}$ is a \mathbb{P}^2 -bundle over a split del Pezzo \mathbb{F}_p -surface of degree 6.

We remark on the side that [Sa, Cor. 2.15] gives $m_p(\underline{Q}(\mathbb{Z}_p)) = |\underline{Q}(\mathbb{Z}/p^r)|/p^{r \dim O}$ more generally for all $r \geq 1$. We have now all the ingredients to give an explicit evaluation of Peyre's empirical formula for the counting function $N(P)$. A combination of (5.10), (5.11), Lemma 18 and Lemma 19 yields

Theorem 8. *Let $X \subset \mathbb{P}^5 \times \mathbb{P}^2 \times \mathbb{P}^2$ be the fourfold defined by (3.4) – (3.6) and let $\tau_H(X)$ be Peyre's adelic Tamagawa volume of $X(\mathbf{A})$ associated to the v -adic norms on ω_X^{-1} in (5.5). Then*

$$\tau_H(X) = 12(\pi^2 + 24 \log 2 - 3) \prod_{\text{all } p} \left(1 - \frac{1}{p} \right)^5 \left(1 + \frac{5}{p} + \frac{6}{p^2} + \frac{5}{p^3} + \frac{1}{p^4} \right).$$

Combining this with (5.1), Lemma 5 and (3.7), we confirm that (1.4) agrees with Peyre's prediction.

Remark. It is possible to interpret $\alpha(X)$ as a real analogue of the p -adic convergence factor $|\underline{T}(\mathbb{F}_p)|/p^{\dim T}$. Let $X^\circ(\mathbb{Q}, P) = \{x \in X^\circ(\mathbf{Q}) : (H \circ f)(x) \leq P\}$. Then, if (5.2) holds, we have by partial summation and (3.8) that

$$(5.15) \quad \sum_{x \in X^\circ(\mathbf{Q}, P)} \frac{1}{(H \circ f)(x)} \sim \tau_H(X) \left(\frac{\alpha(X)}{\text{rk Pic}(X)} \right) (\log P)^{\text{rk Pic}(X)} = \tau_H(x) \int_{\Delta(P)} ds$$

where $\Delta(P)$ is the set of all linear forms Λ on $\text{Pic}(X) \otimes \mathbb{R}$ such that $\Lambda([-K_X]) \leq \log P$ and $\Lambda \in C_{\text{eff}}(X)^\vee$. Now let $x \in X$ be the point where all six x_i -coordinates are equal. We may then identify T with the fibre of the torsor $\varphi_O : O \rightarrow X$ over x such that the neutral element of T corresponds to the point in \mathbb{A}^{10} with all coordinates equal to 1. Let $D(P) \subset T(\mathbb{R})$ be the subset where $\min(u, u_1, u_2, u_3, |w_1|, |w_2|, |w_3|) \geq 1$ and where one and hence all $|x_i^3|$ are at most P . Then, as $y_1 = y_2 = y_3$ on $T(\mathbb{R})$ all w_i have the same sign. If we let $D_+(P) \subset D(P)$ be the subset where all $w_i > 0$ and dt be the measure on $T(\mathbb{R})$ in Lemma 17 (see also Lemma 18), we obtain $\int_{D(P)} dt = 2 \int_{D_+(P)} ds$. Furthermore, by Lemma 4(ii) we have that $\int_{D_+(P)} dt = \int_{\Delta(P)} ds$ such that

$$(5.16) \quad \frac{\alpha(X)}{\text{rk Pic}(X)} (\log P)^{\text{rk Pic}(X)} = \frac{1}{2} \int_{D(P)} dt.$$

We may now give an heuristic derivation of the factor $\alpha(X(\mathbb{R}))\mu_\infty(X(\mathbb{R}))$ in Peyre's constant. Let $F(P)$ be the set of all $\mathbf{r} = (u, u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3) \in \mathbb{R}^{10}$ such that

$$\min(u, u_1, u_2, u_3, |w_1|, |w_2|, |w_3|) \geq 1, \quad u_1v_1 + u_2v_2 + u_3v_3 = 0, \quad \max |x_i^3| \leq P$$

with (x_1, \dots, x_6) as in (4.19). Then, by Lemma 11 and (5.16), we see that (5.15) corresponds to the conjecture

$$\sum_{\mathbf{r} \in F(P) \cap \mathbb{Z}^{10}} \frac{1}{\max |x_i^3|} \sim \mu_\infty(X(\mathbb{R})) \int_{D(P)} dt.$$

To motivate this, let us approximate the sum on the left hand side by

$$\int_{F(P)} \frac{du du_1 du_2 du_3 dv_1 dv_2 dv_3 dw_1 dw_2 dw_3}{\max_{1 \leq j \leq 6} |x_j^3| d(u_1v_1 + u_2v_2 + u_3v_3)} = m_\infty(F(P)).$$

Then by Lemma 17 we obtain $m_\infty(F(P)) \sim \mu_\infty(X(\mathbb{R})) \int_{D(P)} dt$ provided that the average contribution to $m_\infty(F(P))$ from the fibres of $\varphi_{O,\infty} : O(\mathbb{R}) \rightarrow X(\mathbb{R})$ is the same as the contribution from the fibre over x .

6. PRELIMINARY UPPER BOUND ESTIMATES

The rest of the paper features analytic techniques, and it is convenient to introduce the following notation. Let $V(P)$ denote the number of integer sextuples (\mathbf{x}, \mathbf{y}) satisfying (1.1) and (1.3) as well as the size condition $|x_j| \leq P, |y_j| \leq P$ ($1 \leq j \leq 3$). Since any rational point counted by $N(P)$ has exactly two representations $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^6$ with coprime coordinates, we conclude by one of Möbius's inversion formulae that

$$(6.1) \quad N(P) = \frac{1}{2} \sum_{d=1}^{\infty} \mu(d) V(P^{1/3}/d).$$

The remainder of this paper is devoted to a proof of the asymptotic relation

$$(6.2) \quad V(P) = P^3 Q_0(\log P) + O(P^{3-\tau})$$

in which Q_0 is a certain real polynomial of degree 4 with leading coefficient

$$(6.3) \quad \frac{1}{2}(\pi^2 + 24 \log 2 - 3) \prod_p \left(1 - \frac{9}{p^2} + \frac{16}{p^3} - \frac{9}{p^4} + \frac{1}{p^6}\right),$$

and τ is a suitable positive real number. Theorem 1 follows easily from (6.1) once (6.2) and (6.3) are established.

The analytic counting procedures in the proof of Theorem 1 will force us to implement a smooth approximation to the domain of counting. We start by deriving two upper bound estimates that help controlling the error in this transition. At the same time, this will illustrate the use of Lemma 7. Our only additional tool is the following simple estimate.

Lemma 20. *Let $A_1, A_2 \geq 1$, and let $u_1, u_2, u_3 \in \mathbb{N}$ be coprime in pairs, with $u_3 \leq A_2$. Then the number of solutions of $u_1v_1 + u_2v_2 + u_3v_3 = 0$ with $v_j \in \mathbb{Z}$ and $|v_1| \leq A_1, |v_2| \leq A_2$ is $O(A_1A_2u_3^{-1})$.*

To see this, note that we have to count solutions of $u_1v_1 + u_2v_2 \equiv 0 \pmod{u_3}$. Choose v_1 . Then one has to solve $u_2v_2 \equiv c \pmod{u_3}$ for some $c \in \mathbb{Z}$. This has $O(1 + A_2u_3^{-1})$ solutions in v_2 .

Let $\mathcal{Z} \subset \{1, 2, \dots, [P]\}$ be a set of Z natural numbers, and let $V^*(P, \mathcal{Z})$ denote the number of solutions of (1.1) counted by $V(P)$ that satisfy $|y_j| \in \mathcal{Z}$ for at least one $j \in \{1, 2, 3\}$. Similarly, let $V_*(P, \mathcal{Z})$ denote the number of solutions of (1.1) counted by $V(P)$ that satisfy $|x_j| \in \mathcal{Z}$ for at least one $j \in \{1, 2, 3\}$.

Lemma 21. *One has $V^*(P, \mathcal{Z}) \ll P^{2+\varepsilon}Z$ and $V_*(P, \mathcal{Z}) \ll P^{2+\varepsilon}Z$.*

Proof. We begin with estimating $V^*(P, \mathcal{Z})$. First observe that by symmetry in the indices 1, 2, 3, it suffices to estimate the number of solutions with $|y_3| \in \mathcal{Z}$. Hence, by Lemma 7,

$$V^*(P, \mathcal{Z}) \leq 3 \sum_{\substack{w_1u_2u_3u \leq P \\ w_2u_1u_3u \leq P \\ w_3u_1u_2u \in \mathcal{Z}}}^* \sum_{\substack{v_1, v_2, v_3: \\ |v_j w_j| \leq P \\ u_1v_1 + u_2v_2 + u_3v_3 = 0}} 1$$

where \sum^* indicates that the summation is subject to the coprimality conditions (4.2). By Lemma 20,

$$V^*(P, \mathcal{Z}) \ll \sum_{\substack{w_1u_2u_3u \leq P \\ w_2u_1u_3u \leq P \\ w_3u_1u_2u \in \mathcal{Z}}} \frac{P^2}{w_1w_2u_3} \ll P^2(\log P)^3 \sum_{w_3u_1u_2u \in \mathcal{Z}} 1.$$

The standard divisor estimate gives $O(P^\varepsilon Z)$ for the last sum, as required.

Now consider $V_*(P, \mathcal{Z})$. By symmetry, this quantity also does not exceed 3 times the number of solutions counted by $V(P)$ with $x_1 \in \mathcal{Z}$. By Lemma 7 and (4.7) we then see that $V_*(P, \mathcal{Z})$ does not exceed 3 times the number of 10-tuples $w_1, w_2, w_3, u, u_1, u_2, u_3, v_1, v_2, v_3$ with $w_j, u_j, u \in \mathbb{N}$, $v_j \in \mathbb{Z}$ satisfying

$$w_1u_2u_3u \leq P, w_2u_1u_3u \leq P, w_3u_1u_2u \leq P, (u_2; u_3) = 1,$$

and $u_1v_1 + u_2v_2 + u_3v_3 = 0$ with $|v_j w_j| \leq P$ and $|v_1 w_1| \in \mathcal{Z}$. This leaves ZP^ε possibilities for v_1, w_1 by a divisor estimate. For given values of u_1, u_2, u_3 , there are $O(P/(w_2u_3))$ possibilities for v_2 with $u_1v_1 + u_2v_2 \equiv 0 \pmod{u_3}$, and this fixes v_3 through the linear equation; here we took advantage of the condition that $(u_2; u_3) = 1$. It follows that

$$V_*(P, \mathcal{Z}) \ll \sum_{\substack{u_2u_3u \leq P \\ w_2u_1u_3u \leq P \\ w_3u_1u_2u \leq P}} \frac{ZP^{1+\varepsilon}}{w_2u_3} \ll ZP^{2+\varepsilon},$$

as required.

Lemma 21 shows that once one of the variables x_j, y_j in (1.1) is restricted to a slim set, then there are few solutions. However, more general slim regions may well contain many integral points. An argument similar to the above shows that there are $\gg P^3$ points on the subvariety defined by (1.1), (1.2), $y_1y_2y_3 \neq 0$ and the additional condition $y_1 = y_2$. By symmetry, one may be tempted to reduce the evaluation of $V(P)$ to counting integral solutions in a cone of the type $|y_1| \leq |y_2| \leq |y_3|$, but the error introduced from multiple counts of the subvarieties $y_1 = y_2$ and $y_2 = y_3$ is not negligible. This will cause extra difficulties later that we bypass with the introduction of a certain partition of unity in (7.8).

7. WEIGHTS AND INTEGRAL KERNELS

In this section, we compile a number of technical results that will be needed in the analytical counting argument. The main topic is Mellin inversion for certain smooth and rough indicator functions. Some of the analysis is routine. However, the two-dimensional Beta type kernels to be discussed in Lemmas 22, 23 and 24 seem to be a new feature in the study of diophantine problems.

7.1. Weight functions. Counting problems are related to characteristic functions on appropriate regions, and the latter are discontinuous in a natural way. One obtains smooth approximations by convolving them with a smooth approximate delta-distribution. The regions of relevance in this paper are intervals, simplices, and a strip near the diagonal in the two-dimensional plane.

The smoothing will be controlled by two parameters δ and Δ . We suppose from now on that $\Delta \in (0, 1]$. All estimates will be uniform in Δ . In the end, we shall choose Δ as a small negative power of P . The role of δ will be described in due course.

The simplest smoothing is that of a sum over residue classes. Choose a smooth function $q : [0, \infty) \rightarrow [0, 1]$ with $q = 0$ on $[0, 1/4] \cup [7/4, \infty)$, and $q(x) + q(1+x) = 1$ for $x \in [0, 1]$. Then $q^{(j)} \ll_j 1$ for all $j \in \mathbb{N}_0$. Also, when $F : \mathbb{N} \rightarrow \mathbb{C}$ is a function with period $D \in \mathbb{N}$, then

$$(7.1) \quad \sum_{r=1}^D F(r) = \sum_{r=1}^{\infty} q(r/D) F(r).$$

We proceed by smoothing the characteristic function of the interval $[0, 1]$, denoted hereafter by f_0 . Let $\varrho_{\Delta} : [0, \infty) \rightarrow [0, \infty)$ be a smooth non-negative function with

$$(7.2) \quad \text{supp}(\varrho_{\Delta}) \subset (1, 1 + \Delta)$$

and

$$(7.3) \quad \int_0^{\infty} \varrho_{\Delta}(x) \frac{dx}{x} = 1,$$

and such that

$$(7.4) \quad \varrho_{\Delta}^{(j)}(x) \ll_j \Delta^{-1-j}$$

holds for all $j \in \mathbb{N}_0$. For $x \in (0, \infty)$, define

$$(7.5) \quad f_{\Delta}(x) = \int_0^{\infty} \varrho_{\Delta}(z) f_0\left(\frac{x}{z}\right) \frac{dz}{z} = \int_x^{\infty} \varrho_{\Delta}(z) \frac{dz}{z}.$$

It follows from (7.2), (7.3) and (7.4) that

$$(7.6) \quad 0 \leq f_{\Delta}(x) \leq 1 \text{ for all } x \in [0, \infty), \quad f_{\Delta} = 1 \text{ on } [0, 1], \quad \text{supp}(f_{\Delta}) \subset [0, 1 + \Delta],$$

and that

$$(7.7) \quad f_{\Delta}^{(j)} \ll_j \Delta^{-j}$$

holds for all $j \in \mathbb{N}_0$. We also note that $\text{supp}(f'_{\Delta}) \subset [1, 1 + \Delta]$. Thus, f_{Δ} is indeed a smooth approximation to f_0 .

For $n \in \mathbb{N}$ let

$$\mathcal{Q} = \mathcal{Q}(n) = \{\mathbf{x} \in \mathbb{R}^n : x_j \geq 0 \text{ for } 1 \leq j \leq n\}$$

denote the positive quadrant. Our next aim is to construct a certain smooth partition of unity of \mathcal{Q} . For $0 \leq \delta < 1/10$, consider the (infinite) simplex

$$\mathcal{T}_{\delta} = \{\mathbf{x} \in \mathcal{Q} : x_1 \leq (1 + \delta)x_2 \leq \dots \leq (1 + \delta)^{n-1}x_n\}.$$

For $n \geq 2$, define the function $h_{\delta} : \mathcal{Q} \rightarrow [0, 1]$ by

$$h_{\delta}(\mathbf{x}) = f_{\delta}(x_1/x_2) \cdots f_{\delta}(x_{n-1}/x_n)$$

provided that $x_2 \cdots x_n \neq 0$, and put $h_{\delta}(\mathbf{x}) = 0$ otherwise. It is easy to see that h_{δ} vanishes on $\mathcal{Q} \setminus \mathcal{T}_{\delta}$, and $h_{\delta}(\mathbf{x}) = 1$ if $\mathbf{x} \in \mathcal{T}_0$.

The group S_n acts on \mathcal{Q} by permuting coordinates. For $\pi \in S_n$ and $\mathbf{x} \in \mathcal{Q}$ define

$$(7.8) \quad h_{\pi, \delta}(\mathbf{x}) = \frac{h_\delta(\pi(\mathbf{x}))}{\sum_{\sigma \in S_n} h_\delta(\sigma(\mathbf{x}))}.$$

Note that the denominator is between 1 and $n!$ by construction of h_δ . Clearly,

$$(7.9) \quad \sum_{\pi \in S_n} h_{\pi, \delta} = 1 \text{ on } \mathcal{Q} \quad \text{and} \quad h_{\pi, \delta} = 0 \text{ on } \mathcal{Q} \setminus \pi(\mathcal{T}_\delta).$$

The function $h_{\pi, 0}$ is simply the characteristic function on $\pi(\mathcal{T})$.

In our later work, we will use the functions $h_{\pi, \delta}$ only for $\delta = 0$ and one specific positive value of δ . It is therefore not necessary to keep track of the dependence of implicit constants on δ . Hence, from now on, *implied constants may depend on δ* .

For $\delta > 0$, the function h_δ is smooth on \mathcal{Q} , and it is a simple exercise using (7.6) to show that any fixed $(\nu_1, \dots, \nu_n) \in \mathbb{N}_0^n$, the estimate

$$\frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \cdots \frac{\partial^{\nu_n}}{\partial x_n^{\nu_n}} h_\delta(\mathbf{x}) \ll \prod_{j=1}^n x_j^{-\nu_j}$$

holds uniformly in the range $x_j > 0$ ($1 \leq j \leq n$). Consequently, for the same values of \mathbf{x} ,

$$(7.10) \quad \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \cdots \frac{\partial^{\nu_n}}{\partial x_n^{\nu_n}} h_{\pi, \delta}(\mathbf{x}) \ll \prod_{j=1}^n x_j^{-\nu_j}.$$

Now we specialize to $n = 3$. For $\pi \in S_3$ and $\mathbf{x} \in \mathcal{Q}(3)$ we let

$$(7.11) \quad f_{\pi, 0, \delta}(\mathbf{x}) = f_0(x_1)f_0(x_2)f_0(x_3)h_{\pi, \delta}(\mathbf{x}).$$

Then, $f_{\pi, 0, 0}$ is the characteristic function on the tetrahedron $0 \leq x_{\pi(1)} \leq x_{\pi(2)} \leq x_{\pi(3)} \leq 1$. The corresponding smooth version is defined by

$$(7.12) \quad \begin{aligned} f_{\pi, \Delta, \delta}(\mathbf{x}) &= \int_{\mathcal{Q}(3)} \varrho_\Delta(z_1)\varrho_\Delta(z_2)\varrho_\Delta(z_3)f_{\pi, 0, \delta}\left(\frac{x_1}{z_1}, \frac{x_2}{z_2}, \frac{x_3}{z_3}\right) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \frac{dz_3}{z_3} \\ &= \int_{x_3}^\infty \int_{x_2}^\infty \int_{x_1}^\infty \varrho_\Delta(z_1)\varrho_\Delta(z_2)\varrho_\Delta(z_3)h_{\pi, \delta}\left(\frac{x_1}{z_1}, \frac{x_2}{z_2}, \frac{x_3}{z_3}\right) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \frac{dz_3}{z_3}. \end{aligned}$$

It is immediate from (7.9), (7.11) and (7.5) that

$$(7.13) \quad \sum_{\pi \in S_3} f_{\pi, \Delta, \delta}(\mathbf{x}) = f_\Delta(x_1)f_\Delta(x_2)f_\Delta(x_3).$$

Note that the right hand side (and hence the left hand side) of (7.13) is independent of δ . By (7.8) and the last expression in (7.12) we see that

$$(7.14) \quad \text{supp}(f_{\pi, \Delta, \delta}) \subset \{\mathbf{x} \in \mathcal{Q}(3) \mid x_{\pi(1)} \leq \gamma x_{\pi(2)} \leq \gamma^2 x_{\pi(3)} \leq \gamma^3\}.$$

where $\gamma = (1 + \delta)(1 + \Delta)$. Moreover, for $\delta > 0$, the function $f_{\pi, \Delta, \delta}(\mathbf{x})$ is smooth for each $\pi \in S_3$ and satisfies the crude bound

$$(7.15) \quad \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \frac{\partial^{\nu_2}}{\partial x_2^{\nu_2}} \frac{\partial^{\nu_3}}{\partial x_3^{\nu_3}} f_{\pi, \Delta, \delta}(\mathbf{x}) \ll \Delta^{-(\nu_1 + \nu_2 + \nu_3)}$$

for any fixed $(\nu_1, \nu_2, \nu_3) \in \mathbb{N}_0^3$, as can be seen from (7.10) and the last expression in (7.12).

Finally let $k_0^+, k_0^- : \mathcal{Q}(2) \rightarrow [0, 1]$ be the characteristic functions on the sets

$$(7.16) \quad \{\mathbf{x} \in \mathcal{Q}(2) : x_1 + x_2 \leq 1\}, \quad \text{resp.} \quad \{\mathbf{x} \in \mathcal{Q}(2) : x_1 \leq 10, x_2 \leq 10, |x_1 - x_2| \leq 1\}.$$

Note that the region $|x_1 - x_2| \leq 1$ has infinite intersection with $\mathcal{Q}(2)$, therefore we need an additional truncation. Define the smooth functions

$$(7.17) \quad k_\Delta^\pm(x_1, x_2) = \int_{\mathcal{Q}(2)} \varrho_\Delta(z_1)\varrho_\Delta(z_2)k_0^\pm\left(\frac{x_1}{z_1}, \frac{x_2}{z_2}\right) \frac{dz_1}{z_1} \frac{dz_2}{z_2}.$$

As before one then finds that for any fixed ν_1, ν_2 one has

$$(7.18) \quad \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \frac{\partial^{\nu_2}}{\partial x_2^{\nu_2}} k_{\Delta}^{\pm}(\mathbf{x}) \ll \Delta^{-(\nu_1 + \nu_2)}.$$

It is also clear that

$$(7.19) \quad k_{\Delta}^{+}(\mathbf{x}) = 1 \text{ if } x_1 + x_2 \leq 1, \quad \text{supp}(k_{\Delta}^{+}) \subset \{\mathbf{x} \in \mathcal{Q}(2) : x_1 + x_2 \leq 1 + \Delta\}$$

and

$$(7.20) \quad \begin{aligned} k_{\Delta}^{-}(\mathbf{x}) &= 1 \quad \text{if } |x_1 - x_2| \leq 1 - 10\Delta, \ x_1, x_2 \leq 10, \\ \text{supp}(k_{\Delta}^{-}) &\subset \{\mathbf{x} \in \mathcal{Q}(2) : |x_1 - x_2| \leq 1 + 10\Delta, \ x_1, x_2 \leq 10(1 + \Delta)\}. \end{aligned}$$

7.2. Mellin inversion. We begin with a short summary of well-known facts concerning Mellin transforms and the related inversion theorem in a multidimensional set-up. For $\mathbf{s} \in \mathbb{C}^n$ and $\mathbf{x} \in \mathcal{Q}$ write $\mathbf{x}^{\mathbf{s}} = x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}$ in the interest of brevity, and put $\mathbf{1} = (1, 1, \dots, 1)$. A function $g : \mathcal{Q} \rightarrow \mathbb{C}$ is *piecewise continuous* if it is continuous everywhere on \mathcal{Q} except for a compact part of \mathcal{Q} that is contained in the union of finitely many $(n-1)$ -dimensional submanifolds of \mathbb{R}^n . Whenever $g : \mathcal{Q} \rightarrow \mathbb{C}$ is a piecewise continuous, compactly supported and bounded function and $\mathbf{s} \in \mathbb{C}^n$ with $\text{Re}(s_j) > 0$ ($1 \leq j \leq n$), then the integral

$$\widehat{g}(\mathbf{s}) = \int_{\mathcal{Q}} g(\mathbf{x}) \mathbf{x}^{\mathbf{s}-\mathbf{1}} d\mathbf{x}$$

defines a holomorphic function \widehat{g} . If in addition g is continuous, Mellin's inversion formula asserts that for any $\mathbf{c} \in \mathbb{R}^n$ with $c_j > 0$ ($1 \leq j \leq n$) one has

$$(7.21) \quad g(\mathbf{x}) = \left(\frac{1}{2\pi i} \right)^n \int_{(\mathbf{c})} \widehat{g}(\mathbf{s}) \mathbf{x}^{-\mathbf{s}} d\mathbf{s}.$$

Here and later, $\int_{(\mathbf{c})}$ denotes n -fold integration over the lines $s_j = c_j + it_j, t_j \in \mathbb{R}$.

We consider the Mellin transforms of the weight functions that we defined in the previous section. The following estimates for the smooth versions are almost immediate.

Lemma 22. *Let $j_1, j_2, j_3 \in \mathbb{N}$.*

- (i) *The function \widehat{q} can be extended to an entire function satisfying $\widehat{q}(s) \ll_{j_1} (1 + |s|)^{-j_1}$.*
- (ii) *The function $\widehat{f}_{\Delta}(s)$ is holomorphic in $\text{Re } s > 0$ and satisfies $\widehat{f}_{\Delta}(s) \ll_{j_1} \Delta^{-j_1} |s|^{-j_1}$ in $1/10 < \text{Re } s < 2$.*
- (iii) *For any $\pi \in S_3$ and $\delta > 0$, the function $\widehat{f}_{\pi, \Delta, \delta}$ is holomorphic in $\text{Re } s_j > 0$ and satisfies*

$$\widehat{f}_{\pi, \Delta, \delta}(\mathbf{s}) \ll_{j_1, j_2, j_3} \frac{\Delta^{-j_1 - j_2 - j_3}}{|s_1|^{j_1} |s_2|^{j_2} |s_3|^{j_3}} \quad \text{in } \frac{1}{10} < \text{Re } s_j < 2.$$

- (iv) *The function $(s_1, s_2) \mapsto s_1 s_2 \widehat{k}_{\Delta}^{\pm}(s_1, s_2)$ admits an analytic continuation to $\text{Re } s_j > -1$ and satisfies*

$$\widehat{k}_{\Delta}^{\pm}(s_1, s_2) \ll_{j_1, j_2} \frac{\Delta^{-j_1 - j_2}}{|s_1|^{j_1} |s_2|^{j_2}} \quad \text{in } -1/2 \leq \text{Re } s_j \leq 2, \quad |s_j| \geq 1/10.$$

Proof. This is repeated integration by parts in combination with (7.7), (7.15) and (7.18).

More precise statements are possible for the unsmoothed weight functions.

Lemma 23. (i) *One has $\widehat{f}_0(s) = 1/s$.*

(ii) *For $\pi \in S_3$ one has*

$$\widehat{f}_{\pi, 0, 0}(s_1, s_2, s_3) = \frac{1}{s_{\pi(1)}(s_{\pi(1)} + s_{\pi(2)})(s_{\pi(1)} + s_{\pi(2)} + s_{\pi(3)})}.$$

(iii) The function $(s_1, s_2) \mapsto s_1 s_2 \widehat{k}_0^\pm(s_1, s_2)$ admits an analytic continuation to $\operatorname{Re} s_j > -1$. For fixed s_2 with $\operatorname{Re} s_2 > 0$, the function $s_1 \mapsto \widehat{k}_0^\pm(s_1, s_2)$ is meromorphic in a neighbourhood of $s_1 = 0$. At $s_1 = 0$ there is a simple pole with

$$\operatorname{res}_{s_1=0} \widehat{k}_0^\pm(s_1, s_2) = \frac{1}{s_2}.$$

Moreover, $\widehat{k}_0^\pm(s_1, s_2) = \widehat{k}_0^\pm(s_2, s_1)$.

Proof. The proof of (i) and (ii) is a straightforward calculation. Next, observe that the integral representation of the Euler Beta-function [GR, 8.380.1] yields

$$(7.22) \quad \widehat{k}_0^+(s_1, s_2) = \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(1+s_1+s_2)}.$$

Hence the claims in (iii) for k_0^+ are a consequence of elementary properties of the Gamma function. For k_0^- , we need to work directly from the definition. When $\operatorname{Re} s_j > 0$ ($j = 1, 2$), Fubini's theorem gives

$$\widehat{k}_0^-(s_1, s_2) = \int_0^{10} x_2^{s_2-1} \int_{\max(0, x_2-1)}^{\min(x_2+1, 10)} x_1^{s_1-1} dx_1 dx_2.$$

Here, in the inner integral, we extend the integration over $[0, 10]$, and subtract the terms added in to correct the error. This artifice produces the identity

$$(7.23) \quad \widehat{k}_0^-(s_1, s_2) = \frac{10^{s_1+s_2}}{s_1 s_2} - \frac{1}{s_1} F(s_2, s_1) - \frac{1}{s_2} F(s_1, s_2)$$

in which

$$F(s_1, s_2) = \int_0^9 (x+1)^{s_1-1} x^{s_2} dx.$$

This integral defines F as a holomorphic function in $s_1 \in \mathbb{C}$ and $\operatorname{Re} s_2 > -1$. Thus, (7.23) provides the desired continuation of $s_1 s_2 \widehat{k}_0^-(s_1, s_2)$. Also, when $\operatorname{Re} s_1 > 0$ is fixed, it follows from (7.23) that $\widehat{k}_0^-(s_1, s_2)$ has a simple pole at $s_2 = 0$. Its residue is the value at $s_2 = 0$ of the function $s_2 \widehat{k}_0^-(s_1, s_2)$, and hence equals

$$\frac{10^{s_1}}{s_1} - F(s_1, 0) = \frac{1}{s_1}.$$

The final symmetry statement is clear. This completes the proof.

Our final lemma will eventually estimate the error when we remove the smoothing at the end of the argument. The proof is rather long and technical and will occupy the rest of this section.

Lemma 24. *Let $0 \leq \eta \leq 1$.*

(i) *For $1/10 < \operatorname{Re} s < 2$ we have*

$$\widehat{f}_\Delta(s) - \widehat{f}_0(s) \ll \Delta^\eta |s|^{\eta-1} \quad \text{and} \quad \max(\widehat{f}_0(s), \widehat{f}_\Delta(s)) \ll |s|^{-1}.$$

(ii) *For $1/10 < \operatorname{Re} s_j < 2$, $\delta > 0$ and $\pi \in S_3$ we have*

$$\widehat{f}_{\pi, \Delta, \delta}(s) - \widehat{f}_{\pi, 0, \delta}(s) \ll \Delta^{3\eta} |s_1 s_2 s_3|^{\eta-1} \quad \text{and} \quad \max(\widehat{f}_{\pi, 0, \delta}(s), \widehat{f}_{\pi, \Delta, \delta}(s)) \ll |s_1 s_2 s_3|^{-1}.$$

(iii) *Fix $-1/2 \leq \alpha \leq 0$. Then in the region*

$$(7.24) \quad \alpha \leq \operatorname{Re} s_j \leq 2, \quad \operatorname{Re} s_1 + \operatorname{Re} s_2 \geq 1/10, \quad |s_j| \geq 1/10$$

we have

$$\widehat{k}_\Delta^\pm(s) - \widehat{k}_0^\pm(s) \ll \frac{\Delta^\eta}{\max(|s_1|, |s_2|)^{1+\alpha-\eta} \min(|s_1|, |s_2|)^{1/2-\alpha}}$$

and

$$\max(\widehat{k}_0^\pm(s), \widehat{k}_\Delta^\pm(s)) \ll \frac{1}{\max(|s_1|, |s_2|)^{1+\alpha} \min(|s_1|, |s_2|)^{1/2-\alpha}}.$$

The bounds in (i), (ii), (iii) remain valid if the functions on the left hand side of the displayed estimates are replaced by a partial derivative.

Proof. The remark on the derivatives follows immediately from Cauchy's integral formula. We launch the proof of (i), (ii) and (iii) with a deduction of the auxiliary bound

$$(7.25) \quad \widehat{\varrho}_\Delta(s) = 1 + O((\Delta|s|)^\eta)$$

that holds for any fixed $0 \leq \eta \leq 1$ uniformly for $1/10 \leq \operatorname{Re} s \leq 2$. Indeed, it follows easily from (7.2) and (7.4) that

$$(7.26) \quad \widehat{\varrho}_\Delta(s) \ll 1.$$

On the other hand, from (7.5) we conclude

$$(7.27) \quad \widehat{f}_\Delta(s) - \widehat{f}_0(s) = \widehat{\varrho}_\Delta(s)\widehat{f}_0(s) - \widehat{f}_0(s) = \frac{1}{s}(\widehat{\varrho}_\Delta(s) - 1),$$

hence

$$(7.28) \quad \widehat{\varrho}_\Delta(s) - 1 = s \int_0^\infty (f_\Delta(x) - f_0(x))x^{s-1}dx \ll |s|\Delta.$$

Clearly (7.26) and (7.28) imply (7.25).

We are now prepared for the main argument. We have already shown part (i), as an inspection of (7.25) – (7.27) shows. For the proof of part (ii), we note that (7.12) yields

$$\widehat{f}_{\pi,\Delta,\delta}(\mathbf{s}) = \widehat{f}_{\pi,0,\delta}(\mathbf{s})\widehat{\varrho}_\Delta(s_1)\widehat{\varrho}_\Delta(s_2)\widehat{\varrho}_\Delta(s_3).$$

Hence by (7.25) it is enough to show

$$(7.29) \quad \widehat{f}_{\pi,0,\delta} \ll |s_1 s_2 s_3|^{-1}.$$

This can be seen as follows. By (7.11) we have

$$f_{\pi,0,\delta}(\mathbf{x}) = F_0(\mathbf{x})H_{\pi,\delta}(\mathbf{x})$$

where $F_0(\mathbf{x}) = f_0(x_1)f_0(x_2)f_0(x_3)$ and $H_{\pi,\delta}(\mathbf{x}) = h_{\pi,\delta}(\mathbf{x})f_1(x_1)f_1(x_2)f_1(x_3)$. It follows easily from (7.10) that the Mellin transform $\widehat{H}_{\pi,\delta}(\mathbf{s})$ is holomorphic in $\operatorname{Re} s_j > 0$ and rapidly decaying on vertical lines. We observe that

$$\widehat{f}_{\pi,0,\delta}(\mathbf{s}) = \frac{1}{(2\pi i)^3} \int_{(\mathbf{c})} \widehat{F}_0(\mathbf{s} - \mathbf{t})\widehat{H}_{\pi,\delta}(\mathbf{t})d\mathbf{t} = \frac{1}{(2\pi i)^3} \int_{(\mathbf{c})} \prod_{j=1}^3 (s_j - t_j)^{-1} \widehat{H}_{\pi,\delta}(\mathbf{t})d\mathbf{t}$$

for $0 < c_j < \operatorname{Re} s_j$, and (7.29) follows easily.

Finally we prove (iii). The convolution formula (7.17) yields $\widehat{k}_\Delta^\pm(\mathbf{s}) = \widehat{k}_0^\pm(\mathbf{s})\widehat{\varrho}_\Delta(s_1)\widehat{\varrho}_\Delta(s_2)$, hence (iii) follows from (7.25), once we can show that the bound

$$(7.30) \quad \widehat{k}_0^\pm(\mathbf{s}) \ll \frac{1}{\max(|s_1|, |s_2|)^{1+\alpha} \min(|s_1|, |s_2|)^{1/2-\alpha}}$$

holds uniformly for all \mathbf{s} in the region defined by (7.24). For \widehat{k}_0^+ , (7.30) follows from (7.22) in combination with Stirling's formula. In the absence of such an explicit formula for \widehat{k}_0^- , we return to (7.23) and observe that for \mathbf{s} in accordance with (7.24), the first summand on the right hand side of this identity does not exceed $10^4|s_1 s_2|^{-1}$, which in turn is bounded by the right hand side of (7.30). By symmetry, it therefore suffices to establish that in the region described by (7.24), one has

$$(7.31) \quad \frac{1}{s_2} F(s_1, s_2) \ll \frac{1}{\max(|s_1|, |s_2|)^{1+\alpha} \min(|s_1|, |s_2|)^{1/2-\alpha}}.$$

We write $s_j = \sigma_j + it_j$ with real numbers σ_j, t_j , and define the functions $\varphi, \omega : (0, \infty) \rightarrow \mathbb{R}$ by

$$\omega(x) = (x+1)^{\sigma_1-1}x^{\sigma_2}, \quad \varphi(x) = t_1 \log(x+1) + t_2 \log x.$$

Then, the definition of F may be rewritten as

$$(7.32) \quad F(s_1, s_2) = \int_0^9 \omega(x) e^{i\varphi(x)} dx,$$

and we observe that

$$(7.33) \quad \overline{F(s_1, s_2)} = F(\bar{s}_1, \bar{s}_2).$$

The bound (7.31) is trivial in the compact part of (7.24) described by $|t_1| \leq 2$, $|t_2| \leq 2$. In the range $|t_1| \leq 2$, $|t_2| \geq 2$ one has $\min(|s_1|, |s_2|) \asymp 1$ and $\max(|s_1|, |s_2|) \asymp |s_2|$. Also, using $\alpha \leq \sigma_j \leq 2$, we see that

$$|F(s_1, s_2)| \leq \int_0^9 \omega(x) dx \leq \int_0^9 (x+1)x^{\sigma_2} dx \ll 1,$$

and (7.31) follows.

It remains to discuss the case where $|t_1| \geq 2$. Here the treatment must be based on integration by parts, enhanced by a stationary phase argument when necessary. Whenever $0 < a \leq b \leq 9$ and φ' does not vanish on the interval $[a, b]$, partial integration with respect to the phase φ yields the identity

$$(7.34) \quad i \int_a^b \omega(x) e^{i\varphi(x)} dx = \frac{\omega(b)}{\varphi'(b)} e^{i\varphi(b)} - \frac{\omega(a)}{\varphi'(a)} e^{i\varphi(a)} - \int_a^b \left(\frac{\omega'(x)}{\varphi'(x)} - \frac{\omega(x)\varphi''(x)}{\varphi'(x)^2} \right) e^{i\varphi(x)} dx.$$

Now consider the situation where $2 \leq |t_1| \leq \frac{11}{10}|t_2|$. By (7.33), we may assume that $t_2 \geq 2$. Then, for $0 < x \leq 9$, one has

$$(7.35) \quad x\varphi'(x) = t_2 + \frac{t_1 x}{x+1} \geq \frac{t_2}{100}.$$

In particular, $x\varphi'(x)$ is continuous at $x = 0$, and bounded below. By (7.32) and (7.34) with $a \rightarrow 0$, we then have

$$(7.36) \quad iF(s_1, s_2) = \frac{\omega(9)}{\varphi'(9)} e^{i\varphi(9)} - \int_0^9 \left(\frac{x\omega'(x)}{x\varphi'(x)} - \frac{\omega(x) \cdot x^2\varphi''(x)}{(x\varphi'(x))^2} \right) e^{i\varphi(x)} dx.$$

However, for $0 < x \leq 9$, $\alpha \leq \sigma_j \leq 2$ one has $\omega(x) \ll x^{\sigma_2}$, and a short calculation also confirms that

$$x\omega'(x) \ll x^{\sigma_2}, \quad x^2\varphi''(x) = -t_2 - t_1 \left(\frac{x}{x+1} \right)^2 \ll t_2.$$

It is now immediate that the integral on the right hand side of (7.36) is $O(t_2^{-1})$, and consequently, the left hand side of (7.31) is $O(|s_2|^{-2})$, which is more than is required.

By (7.33), the only case that remains to be discussed is when $t_1 > 2$ and $|t_2| \leq \frac{10}{11}t_1$. We write

$$\eta = \frac{1 + |t_2|^{1/2}}{8t_1}, \quad \xi = \frac{-t_2}{t_1 + t_2},$$

and note that $0 < \eta < \frac{1}{4}$. If in addition to the current assumptions one has $t_2 \geq 0$, then by (7.35) we have $\varphi'(x) \geq t_1/10$ throughout the range $0 < x \leq 9$. Hence, we may take $a = 4\eta$, $b = 9$ in (7.34) and, subject to (7.24), estimate by brute force to establish the bound

$$(7.37) \quad \int_{4\eta}^9 \omega(x) e^{i\varphi(x)} dx \ll \frac{\eta^\alpha}{t_1} + \frac{1}{t_1^2} \int_{4\eta}^9 \omega(x) |\varphi''(x)| dx \ll \frac{\eta^\alpha}{t_1} + \frac{\eta^{\alpha-1}t_2}{t_1^2}.$$

Also, by direct estimates,

$$(7.38) \quad \int_0^{4\eta} \omega(x) e^{i\varphi(x)} dx \ll \int_0^{4\eta} x^{\sigma_2} dx \ll \eta^{1+\alpha}.$$

By (7.32), this combines to

$$F(\mathbf{s}) \ll \eta^\alpha \left(\frac{1}{t_1} + \frac{t_2}{\eta t_1^2} + \eta \right) \ll \left(\frac{1 + t_2^{1/2}}{t_1} \right)^{1+\alpha},$$

and (7.31) is immediate.

In the case where $t_2 < 0$ the function φ has exactly one zero on $[0, \infty)$, at $x = \xi$. In the peculiar situation where $\eta > \frac{1}{2}\xi$ the previous argument needs little adjustment: (7.38) remains valid as it stands, and the identity

$$(7.39) \quad x\varphi'(x) = \frac{t_1 + t_2}{1 + x}(x - \xi)$$

shows that for $x \geq 4\eta$ one has $x\varphi'(x) \geq \frac{1}{110}t_1(x - \xi) \geq \frac{1}{220}t_1x$, which yields (7.37), and (7.31) follows as before.

We may now suppose that $\eta \leq \frac{1}{2}\xi$. Also, the currently active constraints that $0 \leq -t_2 \leq \frac{10}{11}t_1$ and $t_1 \geq 2$ imply that $0 \leq \xi \leq 10$. We partition the interval $[0, 9]$ into the three disjoint set

$$J_1 = [0, 9] \cap (-\infty, \xi - \eta], \quad J_2 = [0, 9] \cap (\xi - \eta, \xi + \eta), \quad J_3 = [0, 9] \cap [\xi + \eta, \infty).$$

Then, one routinely finds that

$$\int_{J_2} \omega(x) e^{i\varphi(x)} dx \ll \int_{J_2} x^{\sigma_2} dx \ll \eta \xi^\alpha \ll \frac{|t_2|^{1/2+\alpha}}{t_1^{1+\alpha}}.$$

Next, we note that for $x \in J_1$, one finds from (7.39) the lower bound $x|\varphi'(x)| \geq \frac{1}{110}t_1(\xi - x)$. Thus, by (7.34) with $a \rightarrow 0$ and $b = \min(9, \xi - \eta)$,

$$\begin{aligned} \int_{J_1} \omega(x) e^{i\varphi(x)} dx &\ll \left| \frac{b\omega(b)}{b\varphi'(b)} \right| + \int_0^b \left| \frac{x\omega'(x)}{x\varphi'(x)} \right| dx + \int_0^b \frac{x^2\omega(x)|\varphi''(x)|}{(x\varphi'(x))^2} dx \\ &\ll \frac{\xi^{\alpha+1}}{t_1\eta} + \frac{1}{t_1} \int_0^b \frac{x^{\sigma_2}}{\xi - x} dx + \frac{1}{t_1^2} \int_0^b \frac{x^{\sigma_2}(t_1x^2 + |t_2|)}{(\xi - x)^2} dx \ll \frac{|t_2|^{1/2+\alpha}}{t_1^{1+\alpha}}. \end{aligned}$$

A very similar calculation estimates the integral with J_1 replaced by J_3 to the same precision. By (7.32), the desired estimate (7.31) is now immediate. This completes the proof of Lemma 24.

It should be observed that our method of estimation for \widehat{k}^- applies equally well to \widehat{k}^+ . Thus, the precise information that is contained in (7.22) is, strictly speaking, not required in this paper. On the other hand, Stirling's formula may be used to show that our bound for \widehat{k}^+ is essentially the best possible. Thus the decay of \widehat{k}^+ as $|s_j|$ gets large is far too weak to be in L^1 . This will cause serious technical difficulties in Chapter 9.

8. ANALYTIC METHODS

8.1. Counting with weights. We are now prepared to prove (6.2). The counting function $V(P)$ will be “smoothed” in several steps to facilitate its evaluation by Mellin inversion and Dirichlet series techniques in the following sections. Let $0 < \Delta \leq 1/10$. We begin by writing the definition of $V(P)$ in the form

$$V(P) = \sum_{(\mathbf{x}, \mathbf{y}) \in \mathcal{W}} \prod_{j=1}^3 f_0\left(\frac{|x_j|}{P}\right) f_0\left(\frac{|y_j|}{P}\right),$$

and then replace all $f_0(|y_j|/P)$ by $f_\Delta(|y_j|/P)$. This will produce an error which in view of (7.6) and the notation introduced in the preamble to Lemma 21 is bounded by the quantity $V^*((1 + \Delta)P, [P, (1 + \Delta)P])$. We now invoke the first conclusion in Lemma 21 and apply (7.13) to replace the product $f_\Delta(|y_1|/P)f_\Delta(|y_2|/P)f_\Delta(|y_3|/P)$. For any fixed $\delta \in (0, 1/10)$, this yields

$$(8.1) \quad V(P) = \sum_{\pi \in S_3} \sum_{(\mathbf{x}, \mathbf{y}) \in \mathcal{W}} f_{\pi, \Delta, \delta} \left(\frac{|y_1|}{P}, \frac{|y_2|}{P}, \frac{|y_3|}{P} \right) f_0 \left(\frac{|x_1|}{P} \right) f_0 \left(\frac{|x_2|}{P} \right) f_0 \left(\frac{|x_3|}{P} \right) + O(\Delta P^{3+\varepsilon}).$$

For any $\pi \in S_3$, one has $(\mathbf{x}, \mathbf{y}) \in \mathcal{W}$ if and only if $(\pi\mathbf{x}, \pi\mathbf{y}) \in \mathcal{W}$. Hence, the inner sum on the right hand side of (8.1) is independent of π , so that it suffices to consider henceforth the contribution

from the identical permutation, id. Now insert the parametrization obtained in Lemma 8, and note that the contributions to (8.1) do not depend on the sign of the w_j . This produces the estimate

$$(8.2) \quad V(P) = 48 \sum_{(u, \mathbf{u}, \mathbf{w}) \in \mathbb{N}^7}^* f_{\text{id}, \Delta, \delta} \left(\frac{uu_2u_3w_1}{P}, \frac{uu_1u_3w_2}{P}, \frac{uu_1u_2w_3}{P} \right) \Upsilon(\mathbf{u}, \mathbf{w}) + O(\Delta P^{3+\varepsilon})$$

in which \sum^* indicates the coprimality conditions (4.2) and (4.5), and we wrote

$$\Upsilon(\mathbf{u}, \mathbf{w}) = \sum_{r_1 \in \mathcal{S}(d_1)} \sum_{r_2, r_3 \in \mathbb{Z}} f_0 \left(\frac{|u_2r_3 - u_3r_2|w_1}{P} \right) f_0 \left(\frac{|u_3r_1 - u_1r_3|w_2}{P} \right) f_0 \left(\frac{|u_1r_2 - u_2r_1|w_3}{P} \right).$$

Next, we smooth out the sum $\Upsilon(\mathbf{u}, \mathbf{w})$. The inner sum over r_2, r_3 in the definition of this sum depends only on $r_1 \bmod u_1$. This follows from the concluding remark prior to Lemma 8, for example. Hence, by (7.1), one infers that

$$\Upsilon(\mathbf{u}, \mathbf{w}) = \sum_{r_1=1}^{\infty} \sum_{r_2, r_3 \in \mathbb{Z}} q \left(\frac{r_1}{u_1} \right) f_0 \left(\frac{|u_2r_3 - u_3r_2|w_1}{P} \right) f_0 \left(\frac{|u_3r_1 - u_1r_3|w_2}{P} \right) f_0 \left(\frac{|u_1r_2 - u_2r_1|w_3}{P} \right).$$

Let $\Upsilon^{(1)}(\mathbf{d}, \mathbf{z})$ be the contribution to this sum from terms with $r_2r_3 \neq 0$, let $\Upsilon^{(2)}(\mathbf{d}, \mathbf{z})$ be the contribution from terms with $r_2 = 0, r_3 \neq 0$, and let $\Upsilon^{(3)}(\mathbf{d}, \mathbf{z})$ be the contribution with $r_2 = r_3 = 0$. Then, by symmetry,

$$\Upsilon(\mathbf{u}, \mathbf{w}) = \Upsilon^{(1)}(\mathbf{u}, \mathbf{w}) + 2\Upsilon^{(2)}(\mathbf{u}, \mathbf{w}) + \Upsilon^{(3)}(\mathbf{u}, \mathbf{w}).$$

We rewrite the sums defining $\Upsilon^{(j)}(\mathbf{d}, \mathbf{z})$ by sorting the sum according to the signs of r_2, r_3 . This yields the identities

$$\begin{aligned} \Upsilon^{(1)}(\mathbf{u}, \mathbf{w}) &= \sum_{\mathbf{r} \in \mathbb{N}^3} \sum_{\sigma_2, \sigma_3 \in \{\pm 1\}} q \left(\frac{r_1}{u_1} \right) f_0 \left(\frac{|u_2\sigma_3r_3 - u_3\sigma_2r_2|w_1}{P} \right) f_0 \left(\frac{|u_3r_1 - u_1\sigma_3r_3|w_2}{P} \right) \\ &\quad \times f_0 \left(\frac{|u_1\sigma_2r_2 - u_2r_1|w_3}{P} \right), \\ \Upsilon^{(2)}(\mathbf{u}, \mathbf{w}) &= \sum_{r_1, r_3=1}^{\infty} \sum_{\sigma_3 \in \{\pm 1\}} q \left(\frac{r_1}{u_1} \right) f_0 \left(\frac{u_2r_3w_1}{P} \right) f_0 \left(\frac{|u_3r_1 - u_1\sigma_3r_3|w_2}{P} \right) f_0 \left(\frac{u_2r_1w_3}{P} \right), \\ \Upsilon^{(3)}(\mathbf{u}, \mathbf{w}) &= \sum_{r_1=1}^{\infty} q \left(\frac{r_1}{u_1} \right) f_0 \left(\frac{u_3r_1w_2}{P} \right) f_0 \left(\frac{u_2r_1w_3}{P} \right). \end{aligned}$$

The support conditions of q and $f_{\text{id}, \Delta, \delta}$ in (7.14) imply that non-zero contributions to the sum (8.2) only arise from summands with $1 \leq r_1 \leq 2u_1$ and $uu_2u_3w_1 \leq \frac{5}{4}uu_1u_3w_2 \leq \frac{25}{16}uu_1u_2w_3$. Similarly, since all $(\mathbf{x}, \mathbf{y}) \in \mathcal{W}$ that occur in (8.1) with a non-zero weight satisfy $|x_j| \leq P, |y_j| \leq (1 + \Delta)P$, we deduce from Lemma 9 that $r_iu_jw_k \leq 8P$ holds for any choice of $\{i, j, k\} = \{1, 2, 3\}$. Therefore, by (7.16), we may rewrite expressions like $f_0(|u_2\sigma_3r_3 - u_3\sigma_2r_2|/P)$ in terms of $k_0^{\pm}(u_2r_3/P, u_3r_2/P)$. With

$$E = \{(-, -, -), (+, +, -), (+, -, +), (-, +, +)\}$$

this produces

$$\Upsilon^{(1)}(\mathbf{u}, \mathbf{w}) = \sum_{\mathbf{r} \in \mathbb{N}^3} \sum_{\epsilon \in E} q \left(\frac{r_1}{u_1} \right) k_0^{\epsilon_1} \left(\frac{u_2r_3w_1}{P}, \frac{u_3r_2w_1}{P} \right) k_0^{\epsilon_2} \left(\frac{u_3r_1w_2}{P}, \frac{u_1r_3w_2}{P} \right) k_0^{\epsilon_3} \left(\frac{u_1r_2w_3}{P}, \frac{u_2r_1w_3}{P} \right)$$

and similarly,

$$\Upsilon^{(2)}(\mathbf{u}, \mathbf{w}) = \sum_{r_1, r_3=1}^{\infty} \sum_{\epsilon \in \{\pm\}} q \left(\frac{r_1}{u_1} \right) f_0 \left(\frac{u_2r_3w_1}{P} \right) k_0^{\epsilon} \left(\frac{u_3r_1w_2}{P}, \frac{u_1r_3w_2}{P} \right) f_0 \left(\frac{u_2r_1w_3}{P} \right).$$

We now smooth the sums $\Upsilon^{(j)}(\mathbf{u}, \mathbf{w})$ by replacing f_0 with f_Δ and k_0^\pm with k_Δ^\pm where appropriate. Thus, we define the sums

$$(8.3) \quad \begin{aligned} \Upsilon_\Delta^{(1)}(\mathbf{u}, \mathbf{w}) &= \sum_{\mathbf{r} \in \mathbb{N}^3} \sum_{\epsilon \in E} q\left(\frac{r_1}{u_1}\right) k_\Delta^{\epsilon_1}\left(\frac{u_2 r_3 w_1}{P}, \frac{u_3 r_2 w_1}{P}\right) k_\Delta^{\epsilon_2}\left(\frac{u_3 r_1 w_2}{P}, \frac{u_1 r_3 w_2}{P}\right) k_\Delta^{\epsilon_3}\left(\frac{u_1 r_2 w_3}{P}, \frac{u_2 r_1 w_3}{P}\right), \\ \Upsilon_\Delta^{(2)}(\mathbf{u}, \mathbf{w}) &= \sum_{r_1, r_3=1}^{\infty} \sum_{\epsilon \in \{\pm\}} q\left(\frac{r_1}{u_1}\right) f_\Delta\left(\frac{u_2 r_3 w_1}{P}\right) k_\Delta^\epsilon\left(\frac{u_3 r_1 w_2}{P}, \frac{u_1 r_3 w_2}{P}\right) f_\Delta\left(\frac{u_2 r_1 w_3}{P}\right), \\ \Upsilon_\Delta^{(3)}(\mathbf{u}, \mathbf{w}) &= \sum_{r_1 \geq 1} q\left(\frac{r_1}{u_1}\right) f_\Delta\left(\frac{u_3 r_1 w_2}{P}\right) f_\Delta\left(\frac{u_2 r_1 w_3}{P}\right) \end{aligned}$$

and, in accordance with (8.2),

$$(8.4) \quad V_\Delta^{(j)}(P) = 48 \sum_{(u, \mathbf{u}, \mathbf{w}) \in \mathbb{N}^7}^* f_{\text{id}, \Delta, \delta}\left(\frac{uu_2 u_3 w_1}{P}, \frac{uu_1 u_3 w_2}{P}, \frac{uu_1 u_2 w_3}{P}\right) \Upsilon_\Delta^{(j)}(\mathbf{u}, \mathbf{w}).$$

Then, on recalling (7.14), (7.19) and (7.20) and reversing the above analysis, one finds that the sum $V_\Delta^{(1)}(P) + 2V_\Delta^{(2)}(P) + V_\Delta^{(3)}(P)$ differs from the first term on the right hand side of (8.2) by at most $V_*((1 - 10\Delta)P, [(1 - 10\Delta)P, (1 + 10\Delta)P])$. Here, we have applied the notation used in Lemma 21, and this lemma now shows that

$$(8.5) \quad V(P) = V_\Delta^{(1)}(P) + 2V_\Delta^{(2)}(P) + V_\Delta^{(3)}(P) + O(\Delta P^{3+\varepsilon}).$$

This completes the preparatory transformation of $V(P)$.

8.2. Contour integration. The sums in (8.3) and (8.4) are ready for treatment by Mellin inversion. Shifts of complex contour integrals will ultimately yield asymptotic formulae for $V_\Delta^{(j)}(P)$. The hardest case to analyze will be $j = 1$, and we consider this one first. Let $\mathbf{c} \in (0, \infty)^{10}$. Then, by (8.3), (8.4) and (7.21), one finds that

$$\begin{aligned} V_\Delta^{(1)}(P) &= \sum_{(u, \mathbf{u}, \mathbf{w}, \mathbf{r}) \in \mathbb{N}^{10}}^* \sum_{\epsilon \in E} \frac{48}{(2\pi i)^{10}} \int_{(\mathbf{c})} \frac{P^{s_1+s_2+s_3} \widehat{f}_{\text{id}, \Delta, \delta}(s_1, s_2, s_3)}{w_1^{s_1} w_2^{s_2} w_3^{s_3} u_1^{s_2+s_3} u_2^{s_1+s_3} u_3^{s_1+s_2} u^{s_1+s_2+s_3}} \\ &\quad \frac{P^{s_4+s_5} \widehat{k}_\Delta^{\epsilon_1}(s_4, s_5)}{(r_3 u_2 w_1)^{s_4} (r_2 u_3 w_1)^{s_5}} \frac{P^{s_6+s_7} \widehat{k}_\Delta^{\epsilon_2}(s_6, s_7)}{(r_1 u_3 w_2)^{s_6} (r_3 u_1 w_2)^{s_7}} \frac{P^{s_8+s_9} \widehat{k}_\Delta^{\epsilon_3}(s_8, s_9)}{(r_2 u_1 w_3)^{s_8} (r_1 u_2 w_3)^{s_9}} \frac{\widehat{q}(s_{10}) u_1^{s_{10}}}{r_1^{s_{10}}} ds. \end{aligned}$$

Note that the integral on the right is absolutely convergent. Recall also that \sum^* denotes the coprimality conditions (4.2) and (4.5).

The last identity may be rewritten in more balanced form. With this end in view, we define the function $\Phi_\Delta(\mathbf{s})$ as the sum

$$(8.6) \quad \sum_{\epsilon \in E} \widehat{f}_{\text{id}, \Delta, \delta}\left(\frac{1}{3} + s_1, \frac{1}{3} + s_2, \frac{1}{3} + s_3\right) \widehat{k}_\Delta^{\epsilon_1}\left(\frac{1}{3} + s_4, \frac{1}{3} + s_5\right) \widehat{k}_\Delta^{\epsilon_2}\left(s_6, \frac{2}{3} + s_7\right) \widehat{k}_\Delta^{\epsilon_3}\left(\frac{2}{3} + s_8, s_9\right) \widehat{q}(1 + s_{10}).$$

The dependence of Φ_Δ on δ is suppressed, as on earlier occasions. By Lemma 22(i), (iii) and Lemma 23(iii), the function $s_6 s_9 \Phi_\Delta(\mathbf{s})$ is holomorphic in $\text{Re } s_j > -1/3$. We introduce the linear forms $\ell_j = \ell_j(\mathbf{s})$ by

$$\begin{aligned} \ell_1 &= s_6, & \ell_2 &= s_9, & \ell_3 &= s_6 + s_9 + s_{10}, & \ell_4 &= s_5 + s_8, & \ell_5 &= s_4 + s_7, \\ \ell_6 &= s_1 + s_4 + s_5, & \ell_7 &= s_2 + s_6 + s_7, & \ell_8 &= s_3 + s_8 + s_9, & \ell_9 &= s_2 + s_3 + s_7 + s_8 - s_{10}, \\ \ell_{10} &= s_1 + s_3 + s_4 + s_9, & \ell_{11} &= s_1 + s_2 + s_5 + s_6, & \ell_{12} &= s_1 + s_2 + s_3, \end{aligned}$$

and may then recast the previous expression for $V_{\Delta}^{(1)}(P)$ in the form

$$V_{\Delta}^{(1)}(P) = \sum_{(u, \mathbf{u}, \mathbf{w}, \mathbf{r}) \in \mathbb{N}^{10}}^* \frac{48}{(2\pi i)^{10}} \int_{(\mathbf{c})} \frac{P^{3+s_1+\dots+s_9} s_6 s_9 \Phi_{\Delta}(\mathbf{s})}{\ell_1 \ell_2 r_1^{1+\ell_3} r_2^{1+\ell_4} r_3^{1+\ell_5} w_1^{1+\ell_6} w_2^{1+\ell_7} w_3^{1+\ell_8} u_1^{1+\ell_9} u_2^{1+\ell_{10}} u_3^{1+\ell_{11}} u^{1+\ell_{12}}} ds.$$

Here we may sum inside the integral and are then in a position to apply the theory developed in Chapter 2. Let G be the graph defined in (2.6), and define

$$(8.7) \quad Z(s) = \zeta(1+s)s,$$

$$(8.8) \quad \Theta_{\Delta}(\mathbf{s}) = s_6 s_9 \Phi_{\Delta}(\mathbf{s}) \Xi_G(1 + \ell_6(\mathbf{s}), 1 + \ell_7(\mathbf{s}), \dots, 1 + \ell_{11}(\mathbf{s})) \prod_{j=3}^{12} Z(\ell_j(\mathbf{s})).$$

Note that $\Theta_{\Delta}(\mathbf{s})$ is holomorphic in $|\operatorname{Re} s_j| < 1/10$ by Theorem 5. Hence, when $0 < c_j < 1/10$, the previous expression for $V_{\Delta}^{(1)}(P)$ may be rewritten in the form

$$V_{\Delta}^{(1)}(P) = \frac{48}{(2\pi i)^{10}} \int_{(\mathbf{c})} \frac{P^{3+s_1+\dots+s_9} \Theta_{\Delta}(\mathbf{s})}{\ell_1(\mathbf{s}) \cdot \dots \cdot \ell_{12}(\mathbf{s})} ds.$$

Although a more careful analysis is needed later, for the moment we content ourselves with recording the crude bound

$$(8.9) \quad \Theta_{\Delta}(\mathbf{s}) \ll \Delta^{-99} \prod_{j=1}^{10} (1 + |s_j|)^{-2}$$

that is available uniformly in $|\operatorname{Re} s_j| \leq 1/20$. This follows from Lemma 22 and the convexity bound for Riemann's zeta function in the critical strip, and ensures absolute convergence of all integrals that occur in the following discussion.

We observe that the linear forms ℓ_j , $1 \leq j \leq 12$, span an 8-dimensional vector space. One checks that the linear forms ℓ_1, \dots, ℓ_8 are linearly independent, while

$$\ell_9 = \ell_7 + \ell_8 - \ell_3, \quad \ell_{10} = \ell_6 + \ell_8 - \ell_4, \quad \ell_{11} = \ell_6 + \ell_7 - \ell_5, \quad \ell_{12} = \ell_6 + \ell_7 + \ell_8 - \ell_1 - \ell_2 - \ell_4 - \ell_5.$$

We prepare a linear change of variables $A\mathbf{z} = \mathbf{s}$ where $A \in GL_{10}(\mathbb{R})$ is such that

$$\begin{aligned} z_j &= \ell_j(\mathbf{s}), \quad 1 \leq j \leq 7, \\ z_8 &= \ell_8(\mathbf{s}) + \ell_7(\mathbf{s}) + \ell_6(\mathbf{s}) = s_1 + \dots + s_9, \\ z_9 &= s_1, \quad z_{10} = s_2. \end{aligned}$$

The definition of z_9 and z_{10} is fairly arbitrary, and these variables will play no role in the following computations. A straightforward computation shows that $\det A = 1$.

Now consider the additional linear forms $\lambda_j = \lambda_j(\mathbf{z})$ defined via

$$\lambda_1 = z_6 + z_7 - z_5, \quad \lambda_2 = z_6 + z_7, \quad \lambda_3 = z_6 + z_3, \quad \lambda_4 = z_7 + z_4, \quad \lambda_5 = z_1 + z_2 + z_4 + z_5.$$

We are ready for the change of variable. Let $\eta_1 = 10^{-6}$ and put $c_8 = 5\eta_1$, $c_j = \eta_1$ for $1 \leq j \leq 10$, $j \neq 8$. Then $A\mathbf{c}$ has non-negative coordinates only, as one readily checks, and after a modest computation we conclude that

$$(8.10) \quad V_{\Delta}^{(1)}(P) = \frac{48}{(2\pi i)^{10}} \int_{(\mathbf{c})} \frac{P^{3+z_8} \Theta_{\Delta}(A\mathbf{z}) d\mathbf{z}}{z_1 \cdots z_7 \lambda_1(\mathbf{z}) \cdot \prod_{\nu=2}^5 (z_8 - \lambda_{\nu}(\mathbf{z}))}.$$

For later purposes we unfold the rather compact notation and spell out the integrand in more detail: as one readily checks, one has

$$(8.11) \quad \frac{\Theta_{\Delta}(A\mathbf{z})}{z_1 \cdots z_7 \lambda_1(\mathbf{z}) \prod_{\nu=2}^5 (z_8 - \lambda_{\nu}(\mathbf{z}))}$$

$$\begin{aligned}
&= \sum_{\epsilon \in E} \widehat{f}_{\text{id}, \Delta, \delta} \left(\frac{1}{3} + z_9, \frac{1}{3} + z_{10}, \frac{1}{3} + z_8 - z_1 - z_2 - z_4 - z_5 - z_9 - z_{10} \right) \\
&\quad \times \widehat{k}_{\Delta}^{\epsilon_2} \left(z_1, \frac{2}{3} + z_7 - z_1 - z_{10} \right) \widehat{k}_{\Delta}^{\epsilon_3} \left(\frac{1}{3} + z_1 + z_4 + z_5 + z_9 + z_{10} - z_6 - z_7, z_2 \right) \\
&\quad \times \widehat{k}_{\Delta}^{\epsilon_1} \left(\frac{1}{3} + z_1 + z_{10} + z_5 - z_7, \frac{1}{3} + z_6 + z_7 - z_1 - z_5 - z_9 - z_{10} \right) \widehat{q}(1 + z_3 - z_2 - z_1) H(\mathbf{z})
\end{aligned}$$

where H is a function independent of Δ that satisfies

$$(8.12) \quad H(\mathbf{z}) \ll \prod_{j=1}^{10} (1 + |z_j|)^{1/1000}$$

uniformly in the region $|\text{Re } z_j| \leq 10\eta_1$, $|z_j| \geq 1$. This again follows from standard bounds for Riemann's zeta function close to the line $\text{Re } s = 1$.

8.3. Multiple contour shifts. To extract a main term from (8.10), we apply the common routine and shift the lines of integration of z_1, \dots, z_8 to the left. We will do this step by step. The factor P^{3+z_8} in (8.10) will not be affected until the last step when the line for z_8 will be moved. The following notation will be helpful in describing the manoeuvre. Let $\mathbf{z} \in \mathbb{C}^{10}$ and $I \subset \{1, \dots, 7\}$. Integration over all variables u_i with $i \notin I$ will be denoted by $d\mathbf{z}_I$. We also denote by $\mathbf{z}_I \in \mathbb{C}^{10}$ the 10-tuple whose i -th entry is u_i if $i \notin I$, and equal to 0 if $i \in I$. Furthermore, let

$$\langle \mathbf{z}_I \rangle = \prod_{\substack{1 \leq i \leq 7 \\ i \notin I}} z_i.$$

We begin by shifting the contours of the variables z_1, \dots, z_5 to the left up to $\text{Re } z_j = -7\eta_1$, one after the other. The shift of $\text{Re } z_i = \eta_1$ to $\text{Re } z_i = -7\eta_1$ passes through exactly one pole at $z_i = 0$, and this pole is simple. In order to describe the outcome of the residue theorem, we define for $I \subset \{1, \dots, 5\}$ a vector $\mathbf{c}^I \in \mathbb{R}^{10-|I|}$ by $c_i = \eta_1$ if $i \in \{6, 7, 9, 10\}$, $c_8 = 5\eta_1$ and $c_i = -7\eta_1$ if $i \leq 5$ and $i \notin I$. Then, by (8.10), we infer that

$$V_{\Delta}^{(1)}(P) = \sum_{I \subset \{1, \dots, 5\}} \frac{48}{(2\pi i)^{10-|I|}} \int_{(\mathbf{c}^I)} \frac{P^{3+z_8} \Theta_{\Delta}(A\mathbf{z}_I) d\mathbf{z}_I}{\langle \mathbf{z}_I \rangle \lambda_1(\mathbf{z}_I) \prod_{\nu=2}^5 (z_8 - \lambda_{\nu}(\mathbf{z}_I))}.$$

Next we shift the variables z_6 and z_7 to $\text{Re } z_i = -3\eta_1$. First consider the case where $5 \notin I$. Then $\lambda_1(\mathbf{z}_I) = z_6 + z_7 - z_5$. Each of the two contour shifts passes through exactly one simple pole at $z_j = 0$. Much as before, for a subset $J \subset \{1, 2, 3, 4, 6, 7\}$ let $\tilde{\mathbf{c}}^J \in \mathbb{R}^{10-|J|}$ be defined by $\tilde{c}_j = -7\eta_1$ for $j \leq 5$, $j \notin J$, by $\tilde{c}_j = -3\eta_1$ for $j \in \{6, 7\}$, $j \notin J$, and finally by $\tilde{c}_8 = 5\eta_1$ and $\tilde{c}_j = \eta_1$ for $j \in \{9, 10\}$. Then the contribution of the terms with $5 \notin I$ can be written as

$$(8.13) \quad \sum_{J \subset \{1, 2, 3, 4, 6, 7\}} \frac{48}{(2\pi i)^{10-|J|}} \int_{(\tilde{\mathbf{c}}^J)} \frac{P^{3+z_8} \Theta_{\Delta}(A\mathbf{z}_J) d\mathbf{z}_J}{\langle \mathbf{z}_J \rangle \lambda_1(\mathbf{z}_J) \prod_{\nu=2}^5 (z_8 - \lambda_{\nu}(\mathbf{z}_J))}.$$

The case $5 \in I$ that we now consider, is different. Here $\lambda_1(\mathbf{z}_I) = z_6 + z_7$. Hence, when we shift z_6 to $\text{Re } z_6 = -3\eta_1$, we pass through two simple poles at $z_6 = 0$ and $z_6 = -z_7$. This generates three terms: the residue \mathcal{R}_1 from the pole at $z_6 = 0$, the residue \mathcal{R}_2 from the pole at $z_6 = -z_7$, and a remaining integral \mathcal{I} over the line $\text{Re } z_6 = -3\eta_1$. In \mathcal{R}_1 the variable z_6 is no longer active, and hence, $\lambda_1 = z_7$. Consequently, shifting z_7 to $\text{Re } z_7 = -3\eta_1$ in \mathcal{R}_1 passes through a double pole at $z_7 = 0$ which contributes the residue

$$(8.14) \quad \sum_{\{5, 6, 7\} \subset J \subset \{1, \dots, 7\}} \frac{48}{(2\pi i)^{10-|J|}} \int_{(\tilde{\mathbf{c}}^J)} \frac{P^{3+z_8}}{\langle \mathbf{z}_J \rangle (z_8 - \lambda_3(\mathbf{z}_J))(z_8 - \lambda_5(\mathbf{z}_J))} \frac{\partial}{\partial z_7} \left(\frac{\Theta_{\Delta}(A\mathbf{z}_{\tilde{J}})}{(z_8 - \lambda_2(\mathbf{z}_{\tilde{J}}))(z_8 - \lambda_4(\mathbf{z}_{\tilde{J}}))} \right) \Big|_{z_7=0} d\mathbf{z}_J.$$

Here we have written $\tilde{J} = J \setminus \{7\}$ in the second line to reactivate the variable z_7 for differentiation. After the shift, an integral remains that, together with the contribution of \mathcal{I} , produces a term

identical to (8.13), but where $J \subset \{1, \dots, 7\}$ is subject to the conditions $5, 6 \in J$, $7 \notin J$ or $5 \in J$, $6 \notin J$. Together with (8.13), these terms combine to

$$(8.15) \quad \sum_{\substack{J \subset \{1, \dots, 7\} \\ \{5, 6, 7\} \not\subset J}} \frac{48}{(2\pi i)^{10-|J|}} \int_{(\tilde{c}^J)} \frac{P^{3+z_8} \Theta_{\Delta}(A\mathbf{z}_J) d\mathbf{z}_J}{\langle \mathbf{z}_J \rangle \lambda_1(\mathbf{z}_J) \prod_{\nu=2}^5 (z_8 - \lambda_{\nu}(\mathbf{z}_J))}.$$

The term \mathcal{R}_2 is more complicated. When we shift z_7 , we pass through a double pole at $z_7 = 0$ that contributes the residue

$$(8.16) \quad - \sum_{\substack{J \subset \{1, \dots, 7\} \\ \{5, 6, 7\} \subset J}} \frac{48}{(2\pi i)^{10-|J|}} \int_{(\tilde{c}^J)} \frac{P^{3+z_8}}{\langle \mathbf{z}_J \rangle z_8 (z_8 - \lambda_5(\mathbf{z}_J))} \frac{\partial}{\partial z_7} \left(\frac{\Theta_{\Delta}(A\mathbf{z}_{\tilde{J}})}{(z_8 - \tilde{\lambda}_3(\mathbf{z}_{\tilde{J}}))(z_8 - \lambda_4(\mathbf{z}_{\tilde{J}}))} \right) \Big|_{z_7=0} d\mathbf{z}_J.$$

Here we have written $\tilde{\lambda}_3(\mathbf{z}) = -z_7 + z_3$ and, as before, $\tilde{J} = J \setminus \{7\}$.

Moreover, there may be an additional simple pole at $z_7 = -z_8$, and this is so if and only if $3 \in I$ (that is, $\lambda_2(\mathbf{z}_I) = z_6$). The residue contributes

$$(8.17) \quad - \sum_{\{3, 5, 6, 7\} \subset J \subset \{1, \dots, 7\}} \frac{48}{(2\pi i)^{10-|J|}} \int_{(\tilde{c}^J)} \frac{P^{3+z_8} \Theta_{\Delta}(A\mathbf{z}_J)}{z_8^3 (2z_8 - \lambda_4(\mathbf{z}_J))(z_8 - \lambda_5(\mathbf{z}_J))} d\mathbf{z}_J.$$

After the shift to $\text{Re } z_7 = -3\eta_1$, remaining integral contributes

$$(8.18) \quad - \sum_{\substack{J \subset \{1, \dots, 7\} \\ 5, 6 \in J, 7 \notin J}} \frac{48}{(2\pi i)^{10-|J|}} \int_{(\tilde{c}^J)} \frac{P^{3+z_8} \Theta_{\Delta}(A\mathbf{z}_J) d\mathbf{z}_J}{\langle \mathbf{z}_J \rangle z_7 z_8 (z_8 - \tilde{\lambda}_3(\mathbf{z}_J))(z_8 - \lambda_4(\mathbf{z}_J))(z_8 - \lambda_5(\mathbf{z}_J))}$$

in which again $\tilde{\lambda}_3(\mathbf{z}) = -z_7 + z_3$.

To summarize the above analysis, it is mandatory to record here that $V_{\Delta}^{(1)}(P)$ is the sum of the five terms (8.14) – (8.18). In each term, we finally shift the line of integration $\text{Re } z_8 = \eta_1$ to $\text{Re } z_8 = -\eta_1$. The only pole that may occur is at $z_8 = 0$. Its order depends on the particular set J in the various sums, but it is immediate that no pole of order higher than five does occur. Consequently, the residues of these poles will combine to $P^3 Q_{\Delta}^{(1)}(\log P)$ where Q_{Δ} is a polynomial depending on Δ of degree at most four. Also, the integral over $\text{Re } z_8 = -\eta_1$ that remains after the shift may be estimated by (8.9), and it transpires that its contribution does not exceed $O(P^{3-\eta_1} \Delta^{-99})$. Thus, we have now shown that

$$(8.19) \quad V_{\Delta}^{(1)}(P) = P^3 Q_{\Delta}^{(1)}(\log P) + O(P^{3-\eta_1} \Delta^{-99}).$$

This expansion has an unfortunate defect: the polynomial $Q_{\Delta}^{(1)}$ has coefficients depending on Δ , and this should not be the case. To rectify this, one may replace $\Theta_{\Delta}(A\mathbf{u}_J)$ by $\Theta_0(A\mathbf{u}_J)$ in the computation of all residues at $z_8 = 0$ that contribute to the coefficients of $Q_{\Delta}^{(1)}$. One then obtains a polynomial $Q_0^{(1)}$, say, that is independent of Δ , yet one then has to control the error that is introduced by this procedure. We postpone a detailed discussion of this matter to the next chapter, and continue with the derivation of formulae of the type (8.19) for $V_{\Delta}^{(2)}(P)$ and $V_{\Delta}^{(3)}(P)$.

8.4. Multiple contour shifts II. Having discussed $V_{\Delta}^{(1)}(P)$ in detail, it is now an easy exercise to analyze the less complicated terms $V_{\Delta}^{(2)}(P)$ and $V_{\Delta}^{(3)}(P)$. We can be brief here.

We start with the analysis of $V_{\Delta}^{(2)}(P)$ and recall (8.3). Following the argument at the beginning of Section 8.2, we define

$$\Phi_{\Delta}(\mathbf{s}) = \sum_{\epsilon=\pm} \hat{f}_{\text{id}, \Delta, \delta} \left(\frac{1}{3} + s_1, \frac{1}{3} + s_2, \frac{1}{3} + s_3 \right) \hat{f}_{\Delta} \left(\frac{2}{3} + s_4 \right) \hat{k}_{\Delta}^{\epsilon} \left(\frac{1}{3} + s_5, \frac{1}{3} + s_6 \right) \hat{f}_{\Delta} \left(\frac{2}{3} + s_7 \right) \hat{q}(s_8),$$

the eight linear forms

$$\begin{aligned} \ell_1 &= s_1 + s_2 + s_3, & \ell_2 &= s_2 + s_3 + s_6 - s_8, & \ell_3 &= s_1 + s_2 + s_5, & \ell_4 &= s_1 + s_4, \\ \ell_5 &= s_3 + s_7, & \ell_6 &= s_2 + s_5 + s_6, & \ell_7 &= s_5 + s_7 + s_8, & \ell_8 &= s_4 + s_6 \end{aligned}$$

and the linear polynomial $l(\mathbf{s}) = s_1 + s_3 + s_4 + s_7 + 2$. Note that this overwrites the definitions of Φ and ℓ_j used in the preceding section; confusion should not arise. Then, for any $\mathbf{c} \in (0, 1/4)^8$, we have

$$V_{\Delta}^{(2)}(P) = \sum_{\substack{\mathbf{u}, \mathbf{w} \in \mathbb{N}^3 \\ u, r_1, r_3 \in \mathbb{N}}}^* \frac{48}{(2\pi i)^8} \int_{(\mathbf{c})} \frac{P^{3+s_1+\dots+s_7} \Phi_{\Delta}(\mathbf{s}) d\mathbf{s}}{u^{1+\ell_1(\mathbf{s})} u_1^{1+\ell_2(\mathbf{s})} u_2^{l(\mathbf{s})} u_3^{1+\ell_3(\mathbf{s})} w_1^{1+\ell_4(\mathbf{s})} w_2^{1+\ell_6(\mathbf{s})} w_3^{1+\ell_5(\mathbf{s})} r_1^{1+\ell_7(\mathbf{s})} r_3^{1+\ell_8(\mathbf{s})}}$$

where again \sum^* denotes a summation subject to the coprimality conditions (4.2) and (4.5). We pull the multiple sum inside the integral and obtain the identity

$$V_{\Delta}^{(2)}(P) = \frac{48}{(2\pi i)^8} \int_{(\mathbf{c})} \frac{P^{3+s_1+\dots+s_7} \Theta_{\Delta}(\mathbf{s})}{\ell_1(\mathbf{s}) \cdot \dots \cdot \ell_8(\mathbf{s})} d\mathbf{s}$$

where

$$\Theta_{\Delta}(\mathbf{s}) = \Phi_{\Delta}(\mathbf{s}) \zeta(l(\mathbf{s})) \Xi_G(1 + \ell_2(\mathbf{s}), l(\mathbf{s}), 1 + \ell_3(\mathbf{s}), \dots, 1 + \ell_6(\mathbf{s})) \prod_{j=1}^8 Z(\ell_j(\mathbf{s})),$$

and where Z is still defined by (8.7). Lemma 22 shows that Θ_{Δ} is holomorphic in $|\operatorname{Re} s_j| \leq 1/4$ where the bound

$$\Theta_{\Delta}(\mathbf{s}) \ll \Delta^{-99} \prod_{j=1}^8 (1 + |s_j|)^{-2}$$

holds. This follows as in (8.9). The eight linear forms $\ell_1(\mathbf{s}), \dots, \ell_8(\mathbf{s})$ span a space of dimension 6, and we define the matrix $A \in GL_8(\mathbb{R})$ by $A\mathbf{z} = \mathbf{s}$ and

$$z_j = \ell_j(\mathbf{s}) \quad (1 \leq j \leq 5), \quad z_6 = s_1 + \dots + s_7, \quad z_7 = s_2, \quad z_8 = s_1.$$

One checks that $\det A = 1$. Now choose $\mathbf{c} \in \mathbb{R}^8$ with $c_6 = 3\eta_1$ and $c_j = \eta_1$ for $j \neq 6$. Then, $A\mathbf{c}$ has all entries positive, and a change of variable produces

$$(8.20) \quad V_{\Delta}^{(2)}(P) = \frac{48}{(2\pi i)^8} \int_{(\mathbf{c})} \frac{P^{3+z_6} \Theta_{\Delta}(A\mathbf{z})}{z_1 \dots z_5 (z_6 - z_5 - z_4)(z_6 - z_4 - z_2)(z_6 - z_5 - z_3)} d\mathbf{z}.$$

We are ready to shift contours to the left. The new lines of integration are $\operatorname{Re} z_j = -2\eta_1$ for $1 \leq j \leq 5$, and $\operatorname{Re} z_6 = -\eta_1$. Then, much as before, one obtains

$$(8.21) \quad V_{\Delta}^{(2)}(P) = P^3 Q_{\Delta}^{(2)}(P) + O(P^{3-\eta_1} \Delta^{-99})$$

where $Q_{\Delta}^{(2)}$ is a polynomial depending on Δ of degree at most 2.

The analysis of $V^{(3)}(P)$ is along the same lines. We (re-)define

$$\Phi_{\Delta}(\mathbf{s}) = \widehat{f}_{\text{id}, \Delta, \delta} \left(1 + s_1, \frac{1}{2} + s_2, \frac{1}{2} + s_3\right) \widehat{f}_{\Delta} \left(\frac{1}{2} + s_4\right) \widehat{f}_{\Delta} \left(\frac{1}{2} + s_5\right) \widehat{q}(s_6),$$

the five linear forms

$$\ell_1(\mathbf{s}) = s_2 + s_3 - s_6, \quad \ell_2(\mathbf{s}) = s_1, \quad \ell_3(\mathbf{s}) = s_2 + s_4, \quad \ell_4(\mathbf{s}) = s_3 + s_5, \quad \ell_5(\mathbf{s}) = s_4 + s_5 + s_6$$

and the three linear polynomials

$$l_1(\mathbf{s}) = 2 + s_1 + s_2 + s_3, \quad l_2(\mathbf{s}) = 2 + s_1 + s_3 + s_5, \quad l_3(\mathbf{s}) = 2 + s_1 + s_2 + s_4.$$

Then, for $\mathbf{c} \in (0, 1/4)^6$, one has

$$V_{\Delta}^{(3)}(P) = \sum_{\substack{\mathbf{u}, \mathbf{w} \in \mathbb{N}^3 \\ u, r_1 \in \mathbb{N}}}^* \frac{48}{(2\pi i)^5} \int_{(\mathbf{c})} \frac{P^{3+s_1+\dots+s_5} \Phi_{\Delta}(\mathbf{s}) d\mathbf{s}}{u^{l_1(\mathbf{s})} u_1^{1+\ell_1(\mathbf{s})} u_2^{l_2(\mathbf{s})} u_3^{l_3(\mathbf{s})} w_1^{1+\ell_2(\mathbf{s})} w_2^{1+\ell_3(\mathbf{s})} w_3^{1+\ell_4(\mathbf{s})} r_1^{1+\ell_5(\mathbf{s})}}$$

In $|\operatorname{Re} s_j| \leq 1/4$, a holomorphic function is defined by

$$\Theta_\Delta(\mathbf{s}) = \Phi_\Delta(\mathbf{s}) \Xi_G(1 + \ell_1(\mathbf{s}), \ell_2(\mathbf{s}), \ell_3(\mathbf{s}), 1 + \ell_2(\mathbf{s}), 1 + \ell_3(\mathbf{s}), 1 + \ell_4(\mathbf{s})) \prod_{j=1}^5 Z(\ell_j(\mathbf{s})) \prod_{j=1}^3 \zeta(\ell_j(\mathbf{s})),$$

and the previous formula for $V_\Delta^{(3)}(P)$ can be rewritten as

$$V_\Delta^{(3)}(P) = \frac{48}{(2\pi i)^6} \int_{(\mathbf{c})} \frac{P^{3+s_1+\dots+s_5} \Theta_\Delta(\mathbf{s})}{\ell_1(\mathbf{s}) \cdots \ell_5(\mathbf{s})} d\mathbf{s}.$$

The linear forms ℓ_1, \dots, ℓ_5 span a space of dimension 4, and we define $A \in GL_6(\mathbb{R})$ by $A\mathbf{z} = \mathbf{s}$ and

$$z_j = \ell_j(\mathbf{s}) \quad (1 \leq j \leq 3), \quad z_4 = s_1 + \dots + s_5, \quad z_5 = s_2, \quad z_6 = s_6.$$

Then $\det A = 1$. Choose $\mathbf{c} \in \mathbb{R}^6$ with $c_4 = 3\eta_1$ and $c_j = \eta_1$ for $j \neq 4$. Then $A\mathbf{c}$ has positive coordinates only, and a change of variable yields

$$(8.22) \quad V_\Delta^{(3)}(P) = \frac{48}{(2\pi i)^6} \int_{(\mathbf{c})} \frac{P^{3+z_4} \Theta_\Delta(A\mathbf{z})}{z_1 z_2 z_3 (z_4 - z_3 - z_2)(z_4 - z_2 - z_1)} d\mathbf{z}.$$

One may now move the lines of integration to $\operatorname{Re} z_j = -2\eta_1$ for $1 \leq j \leq 3$, and then to $\operatorname{Re} z_4 = -\eta_1$. An argument similar to the one used above now readily provides the expansion

$$(8.23) \quad V_\Delta^{(3)}(P) = P^3 Q_\Delta^{(3)}(\log P) + O(P^{3-\eta_1} \Delta^{-99})$$

in which $Q_\Delta^{(3)}$ is a polynomial depending on Δ , of degree at most 1. Combining (8.5), (8.19), (8.21) and (8.23), we have shown

$$(8.24) \quad V(P) = P^3 Q_\Delta(\log P) + O(\Delta P^{3+\varepsilon} + P^{3-\eta_1} \Delta^{-99})$$

where

$$(8.25) \quad Q_\Delta = Q_\Delta^{(1)} + 2Q_\Delta^{(2)} + Q_\Delta^{(3)}.$$

9. REMOVING THE SMOOTHING PARAMETER

9.1. A useful lemma. The principal goal in this chapter is to remove the dependence on Δ in the leading term on the right hand side of (8.24). The discussion of this theme generates certain multiple integrals, and we begin with a lemma that ensures the existence of these integrals.

Lemma 25. *Let $m, n \in \mathbb{N}$, and let $\nu = 1/(8n)$. Let $\ell_1, \dots, \ell_m \in \mathbb{R}[x_1, \dots, x_n]$ be m linear forms, and suppose that for any $1 \leq i \leq n$ the coefficient of x_i is non-zero in at least one of the linear forms ℓ_j . Then the function*

$$\prod_{j=1}^n (1 + |x_j|)^{\nu-1} \prod_{j=1}^m (1 + |\ell_j(\mathbf{x})|)^{-1/3}$$

is integrable over \mathbb{R}^n .

Note that the hypotheses on the linear forms ℓ_j cannot be relaxed. For example, if all ℓ_j would be independent of x_1 , then a divergent integral over x_1 would factor off.

For a proof of Lemma 25, write

$$\mathcal{L}(\mathbf{x}) = \prod_{j=1}^m (1 + |\ell_j(\mathbf{x})|).$$

For $1 \leq i \leq n$, the integral

$$I_i = \int_{\mathbb{R}^n} \mathcal{L}(\mathbf{x})^{-4/3} \prod_{\substack{j=1 \\ j \neq i}}^n (1 + |x_j|)^{(3-4n)/(4n-4)} dx$$

certainly exists, because by hypothesis there is a linear form ℓ_l for which the substitution $x_i \mapsto \ell_l$ is non-singular, and the inequality $\mathcal{L}(\mathbf{x}) \geq 1 + |\ell_l(\mathbf{x})|$ shows that the integral

$$\int_{\mathbb{R}^n} (1 + |\ell_l|)^{-4/3} \prod_{\substack{j=1 \\ j \neq i}}^n (1 + |x_j|)^{(3-4n)/(4n-4)} d\ell_l dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

(with integration against x_i omitted) is a majorant. Now, by Hölder's inequality,

$$\int_{\mathbb{R}^n} \mathcal{L}(\mathbf{x})^{-1/3} \prod_{j=1}^n (1 + |x_j|)^{\nu-1} d\mathbf{x} \leq (I_1 I_2 \cdots I_n)^{1/(4n)} \left(\int_{\mathbb{R}^n} \prod_{j=1}^n (1 + |x_j|)^{-1-1/(12n)} d\mathbf{x} \right)^{3/4},$$

which demonstrates the lemma.

9.2. The generic case. We now turn to the main task in this chapter, and derive the desired asymptotic formula (6.2) from (8.24). It will be necessary to compare the polynomial $Q_\Delta(\log P)$ with one that is independent of Δ . We have alluded to this problem already in commentary following (8.19), and the strategy suggested there will now be worked out in detail, in separate sections for the portions $V_\Delta^{(j)}(P)$ contributing to the leading term in (8.24). The goal is to show that there exists a polynomial Q_0 such that

$$(9.1) \quad P^3 Q_\Delta(\log P) = P^3 Q_0(\log P) + O(P^{3+\varepsilon} \Delta^{1/200}).$$

Taking this for granted, it follows from (8.24) that

$$(9.2) \quad V(P) = P^3 Q_0(\log P) + O(P^\varepsilon (\Delta^{1/200} P^3 + P^{3-\eta_1} \Delta^{-99})).$$

One may take $\Delta = P^{-\eta_1/100}$ to infer (6.2) with $\tau = \eta_1/20000$, as required.

We now return to (8.19), and examine the origins of the polynomial $Q_\Delta^{(1)}$. Its coefficients may be computed from the residues at $z_8 = 0$, for each summand in the sums (8.14) – (8.18). Summands corresponding to a set J for which the integrand has no pole at $u_8 = 0$ do not contribute to $Q_\Delta^{(1)}$ and may therefore be ignored.

As an illustrative example, we now examine in full detail the sum (8.15). In this sum, the integrand has a pole of order ν at $z_8 = 0$ if and only if exactly ν of the four linear forms $\lambda_2(\mathbf{z}_J), \dots, \lambda_5(\mathbf{z}_J)$ vanish identically. But $\lambda_l(\mathbf{z}_J)$ vanishes identically if and only if the condition (I) in the list

$$(II) \quad \{6, 7\} \subset J, \quad (III) \quad \{3, 6\} \subset J, \quad (IV) \quad \{4, 7\} \subset J, \quad (V) \quad \{1, 2, 4, 5\} \subset J$$

holds. The condition that $\{5, 6, 7\} \not\subset J$ implies that (II) and (V) cannot hold simultaneously, so that the maximum order of the pole is $\nu = 3$.

We begin with the summand $J = \{3, 6\}$ in (8.15), and write $\hat{J} = J \cup \{8\}$. In this case, the integrand in (8.15) has a simple pole at $z_8 = 0$, the residue of which is

$$(9.3) \quad P^3 \int_{(\hat{\mathbf{c}}^J)} \frac{\Theta_\Delta(A\mathbf{z}_J) d\mathbf{z}_J}{\langle \mathbf{z}_J \rangle \lambda_1(\mathbf{z}_J) \prod_{\nu=2}^5 (z_8 - \lambda_\nu(\mathbf{z}_J))}.$$

Thus, it transpires that the integral here contributes to the constant coefficient in $Q_\Delta^{(1)}$. As suggested earlier, one replaces Θ_Δ by Θ_0 and estimates the error. In the interest of brevity, we write

$$\Psi_\Delta(\mathbf{z}_J) = \frac{\Theta_\Delta(A\mathbf{z}_J)}{\langle \mathbf{z}_J \rangle \lambda_1(\mathbf{z}_J) \prod_{\nu=2}^5 (z_8 - \lambda_\nu(\mathbf{z}_J))}$$

and take $z_3 = z_6 = z_8 = 0$ in (8.11) to infer the alternative representation

$$(9.4) \quad \begin{aligned} \Psi_\Delta(\mathbf{z}_J) &= \widehat{f}_{\text{id}, \Delta, \delta} \left(\frac{1}{3} + z_9, \frac{1}{3} + z_{10}, \frac{1}{3} - z_1 - z_2 - z_4 - z_5 - z_9 - z_{10} \right) \\ &\quad \widehat{k}_\Delta^{\varepsilon_1} \left(\frac{1}{3} + z_1 + z_{10} + z_5 - z_7, \frac{1}{3} + z_7 - z_1 - z_5 - z_9 - z_{10} \right) \widehat{k}_\Delta^{\varepsilon_2} \left(z_1, \frac{2}{3} + z_7 - z_1 - z_{10} \right) \\ &\quad \widehat{k}_\Delta^{\varepsilon_3} \left(\frac{1}{3} + z_1 + z_4 + z_5 + u_9 + z_{10} - z_7, z_2 \right) \widehat{q}(1 - z_1 - z_2) H(\mathbf{z}_J). \end{aligned}$$

Here H is independent of Δ and δ . We may formally take $\Delta = 0$ in (9.4) to define Ψ_0 . Then, the error that arises from replacing Δ by 0 in (9.3) is given by the integral

$$(9.5) \quad P^3 \int_{(\mathfrak{C}^J)} (\Psi_\Delta(\mathbf{z}_j) - \Psi_0(\mathbf{z}_j)) d\mathbf{z}_j$$

To estimate this integral, we parametrize the lines of integration via $z_j = c_j + it_j$, with $\mathbf{t}_j \in \mathbb{R}^7$. Then, by (9.4), Lemma 22 (i), Lemma 24 (with $\alpha = -50\eta_1$) and (8.12), for any $0 < \eta < 1/2$ one finds that

$$(9.6) \quad \Psi_\Delta(\mathbf{u}_j) - \Psi_0(\mathbf{z}_j) \ll \Delta^\eta \mathcal{I}(\mathbf{t}_j)$$

where

$$\begin{aligned} \mathcal{I}(\mathbf{t}_j) = & ((1 + |z_9|)(1 + |z_{10}|)(1 + |z_1 + z_2 + z_4 + z_5 + z_9 + z_{10}|))^{\eta-1} \\ & (1 + |z_1 + z_5 + z_{10} - z_7|)^{\eta-1} (1 + |z_7 - z_1 - z_5 - z_9 - z_{10}|)^{\eta-1/2} \\ & (1 + |z_2|)^{\eta-1} (1 + |z_1 + z_4 + z_5 + z_9 + z_{10} - z_7|)^{\eta-1/2} \\ & (1 + |z_7 - z_1 - z_{10}|)^{\eta-1} (1 + |z_1|)^{\eta-1/2} (1 + |z_2 + z_1|)^{-2} \prod_{j \in \{1,2,4,5,7,9,10\}} (1 + |z_j|)^{1/1000}. \end{aligned}$$

It remains to check that for $\eta = 1/200$, the function $\mathcal{I}(\mathbf{t}_j)$ is integrable over \mathbb{R}^7 . Once this is established, it follows from (9.6) that the expression in (9.5) is bounded by $O(P^3 \Delta^\eta)$, as is required for the verification of (9.1).

The linear change of variable

$$\begin{aligned} x_1 &= t_9, & x_2 &= t_{10}, & x_3 &= t_1 + t_2 + t_4 + t_5 + t_9 + t_{10}, & x_4 &= t_1 + t_5 + t_{10} - t_7 \\ x_5 &= t_2, & x_6 &= t_7 - t_1 - t_{10}, & x_7 &= t_2 + t_1 \end{aligned}$$

has determinant 1, and in the new coordinates, the bound for $\mathcal{I}(\mathbf{t}_j)$ now implies the cruder inequality

$$\mathcal{I}(\mathbf{t}_j) \ll \left(\prod_{j=1}^7 (1 + |x_j|) \right)^{\eta-99/100} ((1 + |x_1 + x_4|)(1 + |x_2 - x_3 + x_6 + x_7|)(1 + |x_5 - x_7|))^{\eta-1/2}.$$

The special case $n = 7$, $m = 3$ of Lemma 25 now shows that when $\eta = 1/200$ the function $\mathcal{I}(\mathbf{t}_j)$ is indeed integrable.

We now consider the remaining terms in (8.15), and begin with a summand corresponding to a set J with $\{3, 6\} \subset J$, but such that none of the conditions (II), (IV), (V) is met. Then, the integrand in (8.15) still has a simple pole at $z_8 = 0$ with residue given by (9.3) where now $\hat{J} = J \cup \{8\}$. The argument following (9.3) remains valid if one puts $z_j = 0$ for $j \in J$, and the error crucial to our present discussion is still given by (9.5). This integral being a lower-dimensional version of the original (9.5), it is apparent from the above analysis that the present (9.5) is again bounded by $O(P^3 \Delta^{1/200})$, as required. Next, consider a set J that occurs in (8.15) with $\{3, 6\} \subset J$, and such that at least one further condition among (II), (IV) and (V) is satisfied. Then, the integrand in (8.15) has a pole of order 2 or 3 at $u_8 = 0$, and its residue is a certain linear combination of

$$(9.7) \quad P^3 \int_{(\mathfrak{C}^J)} \frac{\partial^l}{\partial z_8^l} \frac{\Theta_\Delta(A\mathbf{z}_J)}{\langle \mathbf{z}_J \rangle \lambda_1(\mathbf{z}_J) \prod_{\nu=2}^5 (z_8 - \lambda_\nu(\mathbf{z}_J))} \Big|_{z_8=0} d\mathbf{z}_j$$

with coefficients that are rational polynomials in $\log P$. In order to complete the current programme, we again have to replace Δ by 0 in (9.7) and estimate the error. Following the previous approach, this error is

$$(9.8) \quad P^3 \int_{(\mathfrak{C}^J)} \frac{\partial^l}{\partial z_8^l} (\Psi_\Delta(\mathbf{z}_J) - \Psi_0(\mathbf{z}_J)) \Big|_{z_8=0} d\mathbf{z}_j$$

Since the relevant bounds in Lemma 24 hold for partial derivatives as well, we can estimate the integrand here to the same precision as in (9.6), and it then transpires that again the relevant error is bounded by $O(P^{3+\varepsilon}\Delta^{1/200})$.

This completes the discussion of all terms in (8.15) where (III) holds. Next, we consider the case $J = \{6, 7\}$ and put $\hat{J} = J \cup \{8\}$. Then again, the integrand in (8.15) has a simple pole at $z_8 = 0$ with residue given by (9.3). The dependence on Δ is then removed by the argument following (9.3), but now with the function

$$\begin{aligned}\Psi_\Delta(\mathbf{z}_{\hat{J}}) &= \widehat{f}_{\text{id}, \Delta, \delta} \left(\frac{1}{3} + z_9, \frac{1}{3} + z_{10}, \frac{1}{3} - z_1 - z_2 - z_4 - z_5 - z_9 - z_{10} \right) \\ &\quad \widehat{k}_\Delta^{\varepsilon_3} \left(\frac{1}{3} + z_1 + z_4 + z_5 + z_9 + z_{10}, z_2 \right) \widehat{k}_\Delta^{\varepsilon_1} \left(\frac{1}{3} + z_1 + z_5 + z_{10}, \frac{1}{3} - z_1 - z_5 - z_9 - z_{10} \right) \\ &\quad \widehat{k}_\Delta^{\varepsilon_2} \left(z_1, \frac{2}{3} - z_1 - z_{10} \right) \widehat{q}(1 + z_3 - z_2 - z_1) H(\mathbf{z}_{\hat{J}}).\end{aligned}$$

that is obtained from (8.11) with $z_6 = z_7 = z_8 = 0$. With this new meaning of Ψ_Δ , one has to estimate the integral (9.5). Proceeding as before, one finds that (9.6) still holds with $\mathcal{I}(\mathbf{t}_{\hat{J}})$ now redefined as

$$\begin{aligned}\mathcal{I}(\mathbf{t}_{\hat{J}}) &= ((1 + |z_9|)(1 + |z_{10}|)(1 + |z_1 + z_2 + z_4 + z_5 + z_9 + z_{10}|))^{\eta-1} (1 + |z_1|)^{\eta-1} (1 + |z_1 + z_{10}|)^{\eta-1/2} \\ &\quad (1 + |z_1 + z_5 + z_{10}|)^{\eta-1} (1 + |z_1 + z_5 + z_9 + z_{10}|)^{\eta-1/2} (1 + |z_2|)^{\eta-1} \\ &\quad (1 + |z_1 + z_4 + z_5 + z_9 + z_{10}|)^{\eta-1/2} (1 + |z_1 + z_2 - z_3|)^{-2} \prod_{j \in \{1, 2, 4, 5, 9, 10\}} (1 + |z_j|)^{1/1000}\end{aligned}$$

We take $\eta = 1/200$ and conclude that the integral in (9.5) is $O(P^3\Delta^\eta)$ provided that the function in the previous display is integrable over \mathbb{R}^7 . This follows from Lemma 25, because the linear transformation

$$\begin{aligned}x_1 &= t_9, & x_2 &= t_{10}, & x_3 &= t_1 + t_2 + t_4 + t_5 + t_9 + t_{10}, \\ x_4 &= t_1, & x_5 &= t_1 + t_5 + t_{10}, & x_6 &= t_2 & x_7 &= t_1 + t_2 - t_3\end{aligned}$$

has determinant 1 and shows the above product bounded by

$$(1 + |x_7|)^{-2} ((1 + |x_1|) \cdots (1 + |x_6|))^{\eta-99/100} ((1 + |x_2 + x_4|)(1 + |x_1 + x_5|)(1 + |x_3 - x_6|))^{\eta-1/2}.$$

For sets J with $\{6, 7\} \subset J$ we can modify this argument in much the same way as in the case $\{3, 6\} \subset J$, and obtain the same error estimate.

The next case that we consider is $J = \{4, 7\}$. In this situation, one takes $z_4 = z_7 = z_8 = 0$ in (8.11), and reconsiders the previous error analysis with

$$\begin{aligned}\Psi_\Delta(\mathbf{z}_{\hat{J}}) &= \widehat{f}_{\text{id}, \Delta, \delta} \left(\frac{1}{3} + z_9, \frac{1}{3} + z_{10}, \frac{1}{3} - z_1 - z_2 - z_5 - z_9 - z_{10} \right) \\ &\quad \widehat{k}_\Delta^{\varepsilon_3} \left(\frac{1}{3} + z_1 + z_5 - z_6 + z_9 + z_{10}, z_2 \right) \widehat{k}_\Delta^{\varepsilon_1} \left(\frac{1}{3} + z_1 + z_5 + z_{10}, \frac{1}{3} - z_1 - z_5 + z_6 - z_9 - z_{10} \right) \\ &\quad \widehat{k}_\Delta^{\varepsilon_2} \left(z_1, \frac{2}{3} - z_1 - z_{10} \right) \widehat{q}(1 + z_3 - z_2 - z_1) H(\mathbf{z}_{\hat{J}}).\end{aligned}$$

As before, one verifies (9.6) with

$$\begin{aligned}\mathcal{I}(\mathbf{t}_{\hat{J}}) &= ((1 + |z_9|)(1 + |z_{10}|)(1 + |z_1 + z_2 + z_5 + z_9 + z_{10}|))^{\eta-1} (1 + |z_1|)^{\eta-1} (1 + |z_1 + z_{10}|)^{\eta-1/2} \\ &\quad (1 + |z_1 + z_5 + z_{10}|)^{\eta-1/2} (1 + |z_1 + z_5 - z_6 + z_9 + z_{10}|)^{2\eta-3/2} (1 + |z_2|)^{\eta-1} \\ &\quad (1 + |z_1 + z_2 - z_3|)^{-2} \prod_{j \in \{1, 2, 5, 6, 9, 10\}} (1 + |z_j|)^{1/1000}\end{aligned}$$

Here the encounter the new phenomenon that the argument $z_1 + z_5 - z_6 + z_9 + z_{10}$ occurs twice in the definition of Ψ_Δ . The change of variable

$$\begin{aligned}x_1 &= t_9, & x_2 &= t_{10}, & x_3 &= t_1 + t_2 + t_5 + t_9 + t_{10}, & x_4 &= t_2, \\ x_5 &= t_1, & x_6 &= t_1 + t_5 - t_6 + t_9 + t_{10}, & x_7 &= t_1 + t_2 - t_3\end{aligned}$$

has determinant 1 and shows the previous product bounded by

$$\left(\prod_{j=1}^5 (1 + |x_j|) \right)^{\eta-99/100} (1 + |x_6|)^{2\eta-5/4} (1 + |x_7|)^{-2} (1 + |x_2 + x_4|)^{\eta-1/2} (1 + |x_3 - x_6 - x_1|)^{\eta-1/2}.$$

For $\eta = 1/200$, this product is integrable, as one finds from Lemma 25 with $n = 5$, and the relevant error is therefore bounded as before. Also, it transpires that the more general case where $\{4, 7\} \subset J$ is covered by this argument and the discussion towards the end of case (III) above.

Finally, we turn to case (V) and examine the situation where $J = \{1, 2, 4, 5\}$. Here (8.11) with $z_1 = z_2 = z_4 = z_5 = z_8 = 0$ reduces to

$$\begin{aligned} \Psi_{\Delta}(\mathbf{z}_J) &= \widehat{f}_{\text{id}, \Delta, \delta} \left(\frac{1}{3} + z_9, \frac{1}{3} + z_{10}, \frac{1}{3} - z_9 - z_{10} \right) \text{res}_{u=0}^{\widehat{k}_{\Delta}^{\epsilon_3}} \left(\frac{1}{3} - z_6 - z_7 + z_9 + z_{10}, z \right) \\ &\quad \widehat{k}_{\Delta}^{\epsilon_1} \left(\frac{1}{3} + z_{10} - z_7, \frac{1}{3} + z_6 + z_7 - z_9 - z_{10} \right) \text{res}_{u=0}^{\widehat{k}_{\Delta}^{\epsilon_2}} \left(z, \frac{2}{3} + z_7 - z_{10} \right) \widehat{q}(1 + z_3) H(\mathbf{z}_J). \end{aligned}$$

Following the previous argument, enhanced by Lemma 23(iii), once more, one confirms (9.6) with

$$\begin{aligned} \mathcal{I}(\mathbf{t}_J) &= ((1 + |z_9|)(1 + |z_{10}|)(1 + |z_9 + z_{10}|))^{\eta-1} (1 + |z_6 + z_7 - z_9 - z_{10}|)^{2\eta-3/2} \\ &\quad (1 + |z_7 - z_{10}|)^{\eta-2} (1 + |z_3|)^{-2} \prod_{j \in \{3, 6, 7, 9, 10\}} (1 + |z_j|)^{1/1000}. \end{aligned}$$

This product is integrable, as one finds using the substitution

$$x_1 = t_9, \quad x_2 = t_{10}, \quad x_3 = t_6 + t_7 - t_9 - t_{10}, \quad x_4 = t_7 - t_{10}, \quad x_5 = t_3.$$

Thus the error analysis can be performed as before, and it is again immediate that the more general case $\{1, 2, 4, 5\} \subset J$ is covered by this approach.

This completes the discussion of the term (8.15). An inspection of the terms (8.17) and (8.18) shows that a very similar treatment is possible, and that one does not encounter integrals that need to be checked for existence, other than those examined above. The sums (8.14) and (8.16) contain a derivative in the integrand, and one can handle this in a fashion identical to the treatment of the derivative in (9.8). Then again, one may appeal to the discussion above to conclude, as desired, that the dependence of Δ can be removed with acceptable error. One then arrives at the formula

$$P^3 Q_{\Delta}^{(1)}(\log P) = P^3 Q_0^{(1)}(\log P) + O(P^{3+\epsilon} \Delta^{1/200})$$

where $Q_0^{(1)}$ is a certain polynomial.

9.3. The analysis of $V^{(2)}(P)$ and $V^{(3)}(P)$. The discussion of $V^{(2)}(P)$ is similar to the work in the previous section, but rather less complex. We begin with an inspection of the transition from (8.20) to (8.21). For $J \subset \{1, \dots, 6\}$ and $\mathbf{z} \in \mathbb{C}^8$, let $\mathbf{z}_J \in \mathbb{C}^8$ be the vector that is obtained from \mathbf{z} on replacing z_j by 0 for all $j \in J$. Note that this is in accord with a similar convention in Section 8.3. Also, for $J \subset \{1, \dots, 5\}$ define the vector $\mathbf{c}^J \in \mathbb{R}^{8-|J|}$ by $c_j = -2\eta_1$ for $1 \leq j \leq 5$, $j \notin J$, and by $c_6 = 3\eta_1$, $c_7 = c_8 = \eta_1$, and put

$$\langle \mathbf{z}_J \rangle = \prod_{\substack{1 \leq j \leq 5 \\ j \notin J}} z_j, \quad L(\mathbf{z}) = (z_6 - z_5 - z_4)(z_6 - z_4 - z_2)(z_6 - z_5 - z_3).$$

Then in analogy with (8.13), one finds from (8.20) that

$$(9.9) \quad V_{\Delta}^{(2)}(P) = \sum_{J \subset \{1, \dots, 5\}} \frac{48}{(2\pi i)^{8-|J|}} \int_{(\mathbf{c}^J)} \frac{P^{3+z_6} \Theta_{\Delta}(A\mathbf{z}_J)}{\langle \mathbf{z}_J \rangle L(\mathbf{z}_J)} d\mathbf{z}_J.$$

To derive (8.21), one now shifts the line $\text{Re } z_6 = 3\eta_1$ to $\text{Re } z_6 = -\eta_1$. The integral over $\text{Re } z_6 = -\eta_1$ contributes $O(P^{3-\eta_1} \Delta^{-99})$ by straightforward estimates. The only pole that may occur in the shift

is at $z_6 = 0$, and this will be the case if and only if z_6 divides the polynomial $L(\mathbf{z}_J)$, that is, if one of the conditions

$$(I) \{4, 5\} \subset J, \quad (II) \{2, 4\} \subset J, \quad (III) \{3, 5\} \subset J$$

holds. Summands in (9.9) with this property generate residues at $z_6 = 0$ that assemble to $P^3 Q_\Delta^{(2)}(\log P)$. Following the line of attack in the previous section, we replace Δ by 0 in these residues and estimate the resulting error.

First suppose that $J = \{4, 5\}$ and put $\hat{J} = J \cup \{6\}$. Then the integrand in (9.9) has a simple pole at $z_6 = 0$ with residue

$$(9.10) \quad P^3 \int_{(\tilde{\mathbf{c}}^{\hat{J}})} \frac{\Theta_\Delta(A\mathbf{z}_J)}{z_1 z_2^2 z_3^2} d\mathbf{z}_J;$$

here $(\tilde{\mathbf{c}}^{\hat{J}})$ is the line $\operatorname{Re} z_j = -2\eta_1$ for $1 \leq j \leq 5$, $j \notin J$, and $\operatorname{Re} z_7 = \operatorname{Re} z_8 = \eta_1$. The integrand in (9.10) that we now abbreviate with $\Psi_\Delta(\mathbf{z}_j)$ admits the alternative expression

$$(9.11) \quad \begin{aligned} \Psi_\Delta(\mathbf{z}_j) = & \hat{f}_{\text{id}, \Delta, \delta} \left(\frac{1}{3} + z_8, \frac{1}{3} + z_7, \frac{1}{3} + z_1 - z_7 - z_8 \right) \hat{k}_\Delta^\pm \left(\frac{1}{3} + z_3 - z_7 - z_8, \frac{1}{3} - z_3 + z_8 \right) \\ & \hat{f}_\Delta \left(\frac{2}{3} - z_8 \right) \hat{f}_\Delta \left(\frac{2}{3} - z_1 + z_7 + z_8 \right) \hat{q}(z_1 - z_2 - z_3) H(\mathbf{z}_j) \end{aligned}$$

where H is holomorphic in $|\operatorname{Re} u_j| \leq 1/4$ and satisfies the bound (8.12). This much is obtained in analogy to the argument leading to (9.4). By Lemma 22 (i) and Lemma 24, on the lines of integration parametrized by $z_j = c_j + it_j$, one has

$$\Psi_\Delta(\mathbf{z}_j) - \Psi_0(\mathbf{z}_j) \ll \Delta^\eta \mathcal{I}(\mathbf{t}_j)$$

where $\eta = 1/200$ and

$$\begin{aligned} \mathcal{I}(\mathbf{t}_j) = & ((1 + |z_8|)(1 + |z_7|)(1 + |z_1 - z_7 - z_8|))^{\eta-1} (1 + |z_3 - z_7 - z_8|)^{\eta-1} (1 + |z_3 - z_8|)^{\eta-1/2} \\ & (1 + |z_8|)^{\eta-1} (1 + |z_1 - z_7 - z_8|)^{\eta-1} (1 + |z_1 - z_2 - z_3|)^{-2} \prod_{j \in \{1, 2, 7, 8\}} (1 + |z_j|)^{1/1000}. \end{aligned}$$

The change of variables

$$x_1 = t_8, \quad x_2 = t_7, \quad x_3 = t_1 - t_7 - t_8, \quad x_4 = t_1 - t_2 - t_3, \quad x_5 = t_3 - t_7 - t_8,$$

has determinant 1, and in the new coordinates one can bound $\mathcal{I}(\mathbf{t}_j)$ as

$$((1 + |x_1|)(1 + |x_3|))^{2\eta-199/100} (1 + |x_4|)^{-2} ((1 + |x_2|)(1 + |x_5|))^{\eta-99/100} (1 + |x_5 + x_2|)^{\eta-1/2}.$$

Hence by Lemma 25 one finds that $\mathcal{I}(\mathbf{t}_j)$ is integrable, and consequently that

$$\int_{(\tilde{\mathbf{c}}^{\hat{J}})} (\Psi_\Delta(\mathbf{z}_j) - \Psi_0(\mathbf{z}_j)) d\mathbf{z}_j \ll \Delta^\eta.$$

Therefore one may indeed replace Δ by 0 in (9.10) at the cost of an error not exceeding $O(\Delta^\eta P^3)$.

Little change is necessary for the treatment of summands corresponding to other sets J with $\{4, 5\} \subset J \subset \{1, \dots, 5\}$. For some of these sets, poles of order 2 or 3 at $z_6 = 0$ occur, and it will then be necessary to consider certain partial derivatives with respect to z_6 , similar to the occurrence of derivatives in (9.8). An inspection of the deliberations following (9.8) shows that the current situation is fully covered by the above treatment, and it transpires that for all sets J with $\{4, 5\} \subset J \subset \{1, \dots, 5\}$, the computation of the residue at $z_6 = 0$ of the integrand in (9.9) may be performed with $\Delta = 0$ at the cost of a total error not exceeding $O(P^{3+\varepsilon} \Delta^\eta)$.

Next, consider the case $J = \{2, 4\}$ and follow the same pattern as before. The right hand side of (9.11) is now to be replaced by

$$\begin{aligned} & \hat{f}_{\text{id}, \Delta, \delta} \left(\frac{1}{3} + z_8, \frac{1}{3} + z_7, \frac{1}{3} + z_1 - z_7 - z_8 \right) \hat{k}_\Delta^\pm \left(\frac{1}{3} + z_3 - z_7 - z_8, \frac{1}{3} - z_3 - z_5 + z_8 \right) \\ & \hat{f}_\Delta \left(\frac{2}{3} - z_8 \right) \hat{f}_\Delta \left(\frac{2}{3} - z_1 + z_5 + z_7 + z_8 \right) \hat{q}(z_1 - z_3 - z_5) H(\mathbf{z}), \end{aligned}$$

and one is then led to check integrability of the product

$$\begin{aligned} & ((1 + |z_8|)(1 + |z_7|)(1 + |z_1 - z_7 - z_8|))^{\eta-1} (1 + |z_3 - z_7 - z_8|)^{\eta-1} (1 + |z_3 + z_5 - z_8|)^{\eta-1/2} \\ & (1 + |z_8|)^{\eta-1} (1 + |z_1 - z_5 - z_7 - z_8|)^{\eta-1} (z_1 - z_3 - z_5)^{-2} \prod_{j \in \{1,3,5,7,8\}} (1 + |z_j|)^{1/1000} \end{aligned}$$

which is provided by Lemma 25 after the substitution

$$x_1 = t_8, \quad x_2 = t_7, \quad x_3 = t_1 - t_7 - t_8, \quad x_4 = t_3 - t_7 - t_8, \quad x_5 = t_1 - t_3 - t_5.$$

For the other cases with $\{2, 4\} \subset J \subset \{1, \dots, 5\}$ this argument may be modified along the lines suggested in the penultimate paragraph.

Now consider $J = \{3, 5\}$ where the right hand side of (9.11) should read

$$\begin{aligned} & \widehat{f}_{\text{id}, \Delta, \delta} \left(\frac{1}{3} + z_8, \frac{1}{3} + z_7, \frac{1}{3} + z_1 - z_7 - z_8 \right) \widehat{k}_{\Delta}^{\pm} \left(\frac{1}{3} - z_7 - z_8, \frac{1}{3} - z_4 + z_8 \right) \\ & \widehat{f}_{\Delta} \left(\frac{2}{3} + z_4 - z_8 \right) \widehat{f}_{\Delta} \left(\frac{2}{3} - z_1 + z_7 + z_8 \right) \widehat{q}(z_1 - z_2 - z_4) H(\mathbf{z}), \end{aligned}$$

and one has to consider the product

$$\begin{aligned} & ((1 + |z_8|)(1 + |z_7|)(1 + |z_1 - z_7 - z_8|))^{\eta-1} (1 + |z_7 + z_8|)^{\eta-1} (1 + |z_4 - z_8|)^{\eta-1/2} \\ & (1 + |z_4 - z_8|)^{\eta-1} (1 + |z_1 - z_7 - z_8|)^{\eta-1} (1 + |z_1 - z_2 - z_4|)^{-2} \prod_{j \in \{1,3,5,7,8\}} (1 + |z_j|)^{1/1000}. \end{aligned}$$

The substitution

$$x_1 = t_8, \quad x_2 = t_7, \quad x_3 = t_1 - t_7 - t_8, \quad x_4 = t_4 - t_8, \quad x_5 = t_1 - t_2 - t_4$$

shows that Lemma 25 again yields the desired integrability of $\mathcal{I}(\mathbf{t}_j)$. As before, other cases with $\{3, 5\} \subset J \subset \{1, \dots, 5\}$ are very similar, and we may now conclude that there exists a polynomial $Q_0^{(2)}$ of degree at most 2 and such that

$$P^3 Q_{\Delta}^{(2)}(P) = P^3 Q_0^{(2)}(\log P) + O(P^{3+\varepsilon} \Delta^{1/200}).$$

Finally, we turn our attention to $V_{\Delta}^{(3)}(P)$. Here a brief inspection of (8.23) suffices to confirm that the now familiar procedure again yields a polynomial of degree at most 1 with

$$P^3 Q_{\Delta}^{(3)}(\log P) = P^3 Q_0^{(3)}(\log P) + O(P^{3+\varepsilon} \Delta^{1/200}).$$

We may leave the details to the reader. This completes the proof of (9.1).

10. THE PEYRE CONSTANT

It remains to determine the leading coefficient of the polynomial Q_0 in (9.2) in order to complete the proof of (6.2) and hence of Theorem 1. It suffices to consider $Q_0^{(1)}$, since $Q_0^{(2)}$ and $Q_0^{(3)}$ have smaller degree. An inspection of the terms (8.14) – (8.18) shows that only the terms (8.14) and (8.17) contribute, and only if $J = \{1, \dots, 7\}$. Thus the leading coefficient is given by

$$\begin{aligned} & \sum_{\epsilon \in E} \frac{48}{(2\pi i)^2} \int_{(\eta)} \int_{(\eta)} \frac{2 - \frac{1}{2}}{24} \Theta_0(A(0, \dots, 0, z_9, z_{10})^T) dz_9 dz_{10} \\ & = \sum_{\epsilon \in E} \frac{3}{(2\pi i)^2} \int_{(\eta)} \int_{(\eta)} \Theta_0(z_9, z_{10}, -z_9 - z_{10}; z_{10}, -z_9 - z_{10}; 0, -z_{10}; z_9 + z_{10}, 0; 0) dz_9 dz_{10}. \end{aligned}$$

By (8.6) – (8.8) and Lemma 23 (iii), this equals

$$\sum_{\epsilon \in \{\pm\}} \frac{6 \Xi_G(1, \dots, 1)}{(2\pi i)^2} \int_{(\eta)} \int_{(\eta)} \frac{\widehat{f}_{\pi, 0, \delta} \left(\frac{1}{3} + z_9, \frac{1}{3} + z_{10}, \frac{1}{3} - z_9 - z_{10} \right) \widehat{k}_0^{\epsilon} \left(\frac{1}{3} + z_{10}, \frac{1}{3} - z_9 - z_{10} \right)}{\left(\frac{2}{3} - z_{10} \right) \left(\frac{2}{3} + z_9 + z_{10} \right)} dz_9 dz_{10}.$$

We call this constant C_δ and claim that $C_\delta = C_0$ for all sufficiently small δ . This can be checked by direct computation, but we can also argue as follows. It is clear from (7.11) and (7.8) that $C_\delta \rightarrow C_0$ as $\delta \rightarrow 0$. We have already shown that

$$\frac{V(P)}{P^3(\log P)^4} = C_\delta + O_\delta\left(\frac{1}{\log P}\right)$$

for any $\delta > 0$. Combining these two asymptotic relations we find that

$$\lim_{P \rightarrow \infty} \frac{V(P)}{P^3(\log P)^4} = C_0$$

as claimed. By Lemma 23(ii) it therefore remains to compute

$$C_0 = \frac{6\Xi_G(1, \dots, 1)}{(2\pi i)^2} \int_{(\frac{1}{3}+\eta)} \int_{(\frac{1}{3}+\eta)} \frac{\widehat{k}_0^+(z_{10}, 1 - z_9 - z_{10}) + \widehat{k}_0^-(z_{10}, 1 - z_9 - z_{10})}{z_9(z_9 + z_{10})^2(1 - z_{10})} dz_9 dz_{10}.$$

We make a change of variables $v_1 = 1 - z_{10}$, $v_2 = z_9 + z_{10}$ and insert the definition of \widehat{k}_0^\pm as a double Mellin transform. In this way we see

$$C_0 = 6\Xi_G(1, \dots, 1) \int_{\mathcal{Q}(2)} (k_0^+(\mathbf{x}) + k_0^-(\mathbf{x})) \frac{1}{(2\pi i)^2} \int_{(\frac{2}{3})} \int_{(\frac{2}{3})} \frac{x_1^{-v_1} x_2^{-v_2} d\mathbf{v}}{(v_1 + v_2 - 1)v_1 v_2^2} d\mathbf{x}.$$

Shifting the v_1, v_2 -contours to the right, we see that the inner integral vanishes unless $x_1, x_2 \leq 1$. In this region, however, k_0^- is constantly 1. In particular we see that the non-canonical choices in the definitions (7.16), (7.17) of k_0^- and of q play no role, as it should be. Shifting the v_1, v_2 -contours to the left, one readily computes the inner integral. The constant $\Xi_G(1, \dots, 1)$ has been computed in (2.7), and the rest is a straightforward evaluation of elementary integrals:

$$\begin{aligned} & 6\Xi_G(1, \dots, 1) \int_0^1 \int_0^1 (k_0^+(\mathbf{x}) + k_0^-(\mathbf{x})) \left(\log(x_2) - 1 + \frac{1 + \max(0, \log(x_1/x_2))}{\max(x_1, x_2)} \right) d\mathbf{x} \\ &= 6\Xi_G(1, \dots, 1) \left(\left(-\frac{5}{4} + \frac{\pi^2}{12} + 2\log 2 \right) + 1 \right) = \frac{1}{2}(\pi^2 + 24\log 2 - 3) \prod_p \left(1 - \frac{9}{p^2} + \frac{16}{p^3} - \frac{9}{p^4} + \frac{1}{p^6} \right) \end{aligned}$$

as claimed in (6.3).

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